

The generalized cusp in ABJ(M) $\mathcal{N} = 6$ Super Chern-Simons theories

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ABSTRACT: We construct a generalized cusped Wilson loop operator in $\mathcal{N} = 6$ super Chern-Simons-matter theories which is locally invariant under half of the supercharges. It depends on two parameters and interpolates smoothly between the 1/2 BPS line or circle and a pair of antiparallel lines, representing a natural generalization of the quark-antiquark potential in ABJ(M) theories. For particular choices of the parameters we obtain 1/6 BPS configurations that, mapped on S^2 by a conformal transformation, realize a three-dimensional analogue of the wedge DGRT Wilson loop of $\mathcal{N} = 4$. The cusp couples, in addition to the gauge and scalar fields of the theory, also to the fermions in the bi-fundamental representation of the $U(N) \times U(M)$ gauge group and its expectation value is expressed as the holonomy of a suitable superconnection. We discuss the definition of these observables in terms of traces and the role of the boundary conditions of fermions along the loop. We perform a complete two-loop analysis, obtaining an explicit result for the generalized cusp at the second non-trivial order, from which we read off the interaction potential between heavy 1/2 BPS particles in the ABJ(M) model. Our results open the possibility to explore in the three-dimensional case the connection between localization properties and integrability, recently advocated in $D = 4$.

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1 Introduction and results

The duality between string theory on asymptotically AdS spaces and conformal gauge theories, usually known as the AdS/CFT correspondence, has experienced an important evolution in the last few years. General non-BPS observables, as anomalous dimensions of composite operators and scattering amplitudes, can now be studied at high precision level providing sophisticated tests of the correspondence and sometimes offering non-trivial interpolating functions between weak and strong coupling. In both cases, the underlying integrability properties of the planar theory play a crucial role in the exact quantum evaluation and allow to follow the transition between the opposite regimes. Dramatic progresses have also concerned more traditional investigations, as the study of protected sectors of supersymmetric gauge theories: the introduction of powerful localization techniques makes now possible the exact computation of complicated path-integrals, providing again examples of interpolation between perturbative and asymptotic behaviors. It is tempting to speculate if the two different approaches, integrability and localization, could be somehow connected, at least in the computation of specific observables.

This evocative possibility has been vigorously advocated in [1] for a general class of Wilson loops in $\mathcal{N} = 4$ super Yang-Mills theory and concretely realized in a series of recent papers [2–5], where a new set of integral equations of TBA type, describing exactly a generalized cusp anomalous dimension, have been derived and checked against localization and perturbation theory at three loops. The result is striking and it contains, in principle, the all-order expression of the static potential between two heavy charged particles in four-dimensional maximally supersymmetric gauge theory.

Wilson loops are important observables in nonabelian gauge theories: they compute the potential between heavy colored probes, representing an order parameter for confinement [6] and encode a large part of information of the high-energy scattering between charged particles [7, 8]. In $\mathcal{N} = 4$ super Yang-Mills theory they play a prominent role, being the observables directly related to the fundamental string of the dual theory in $AdS_5 \times S^5$ [9, 10]. They are conjectured to calculate scattering amplitudes exactly [11, 12] and in particular cases they are BPS operators [13, 14], whose quantum expectation value can be derived for any strength of the coupling constant [15]. In [1] a two-parameters family of Wilson loops in $\mathcal{N} = 4$ SYM has been proposed and studied both at weak and strong coupling: it consists of two rays in \mathbb{R}^4 meeting at a point, with a cusp angle denoted by $\pi - \varphi$. Because the Maldacena-Wilson loops in $\mathcal{N} = 4$ SYM couple to a real scalar field, it is natural to consider different scalars on different rays, connected by an R -symmetry rotation of parameter

θ . By varying continuously φ and θ and performing suitable conformal transformations, these observables can be related to important physical quantities and to interesting BPS configurations. Mapping the theory to $S^3 \times \mathbb{R}$ one obtains a pair of antiparallel lines, separated by angle $\pi - \varphi$ on the sphere, and derives the potential $V(\lambda, \varphi, \theta)$ between two heavy W -bosons propagating over a large time T :

$$\langle \mathcal{W}_{\text{lines}} \rangle \simeq \exp(-TV(\lambda, \varphi, \theta)). \quad (1.1)$$

The usual potential in the flat space is easily recovered by taking the residue of $V(\lambda, \varphi, 0)$ as $\varphi \rightarrow \pi$ [1]. In the cusped version, the Wilson loop has the leading form

$$\langle \mathcal{W}_{\text{cusp}} \rangle \simeq \exp\left(-\Gamma_{\text{cusp}}(\lambda, \varphi, \theta) \log\left(\frac{\Lambda}{\epsilon}\right)\right), \quad (1.2)$$

and it turns out that $\Gamma_{\text{cusp}}(\lambda, \varphi, \theta) = V(\lambda, \varphi, \theta)$, with Λ and ϵ being IR and UV cut-offs respectively [16, 17]. The cusped Wilson loops are strictly related to scattering amplitudes in Minkowski space: taking φ imaginary and large, $\varphi = i\phi$, produces to the so-called universal cusp anomalous dimension $\gamma_{\text{cusp}}(\lambda)$ [18]

$$\lim_{\phi \rightarrow \infty} \Gamma_{\text{cusp}}(\lambda, i\varphi, \theta) = \frac{\varphi \gamma_{\text{cusp}}(\lambda)}{4}. \quad (1.3)$$

Remarkably BPS configurations are also included in the family: for $\theta = \pm\varphi$ the cusped Wilson loop is of Zarembo's type [19], implying the vanishing of $\Gamma_{\text{cusp}}(\lambda, \varphi, \theta)$ in this case. By mapping conformally the rays into S^2 we recover instead the DGRT wedge, a well studied 1/4 BPS operator [20–22] belonging to the general class of loops on S^3 introduced in [20]. The quantum expectation value of the wedge is computed exactly by two-dimensional Yang-Mills theory [21, 22], a property shared with other DGRT loops on S^2 [23]. We remark that path-integral localization properties are essential in order to derive these results.

In the limit of small φ , $\Gamma_{\text{cusp}}(\lambda, \varphi, 0) \simeq -B(\lambda)\varphi^2$ computes also the radiation of a particle moving along an arbitrary smooth path [2] and an exact expression can be obtained by exploiting the BPS properties at $\varphi = 0$. A simple modification applies as well in expanding around the general BPS points $\theta = \pm\varphi$, thanks to the knowledge of the all-orders expression of the DGRT wedge. Further, in [4, 5], a powerful set of TBA type integral equations for the generalized cusp was derived, using integrability: the explicit one-loop perturbative result [4] and the three-loop expression of the near BPS limit [3] were recovered as a check of the procedure.

In view of these recent and exciting developments, it appears natural to wonder whether similar results could also be obtained for other superconformal gauge theories with integrable structures: the obvious choice is to investigate what happens in $\mathcal{N} = 6$ super Chern-Simons theories with matter, also known as ABJ(M) [24, 25]. Wilson loops in ABJ(M) theory are still a rather unexplored subject: equivalence with scattering amplitudes has been shown at the second order in perturbation theory [26–29] and a quite mysterious functional similarity with their four-dimensional cousins seems to emerge. On the other hand, supersymmetric

configurations, supported on straight lines and circles, have been discovered [30–33] and a localization formula, reducing the computation to an explicit matrix model average, has been derived [35]. Concrete results at weak and strong coupling, using matrix model and topological string techniques, are presented in a beautiful series of papers [36–39], where also various aspects of the partition function on S^3 have been discussed. Nevertheless many issues should be understood in order to extend the generalized cusp program. First of all it does not even exist a computation of the standard quark-antiquark potential nor of the conventional cusped Wilson loop. Secondly, BPS lines and circles appear in two fashions, distinguished by the degree of preserved supersymmetry: we have the 1/6 BPS operators, originally defined in [31, 32] (see also [33]), that are a straightforward extension of the Maldacena-Wilson loop to the three-dimensional case. The (quadratic) coupling with the scalar fields of the theory is governed by a mass matrix M_J^I , preserving an $SU(2) \times SU(2)$ of the original R -symmetry, while gauge fields appear in the usual way. Although its simplicity, this kind of loop cannot be considered the dual of the fundamental string living in $AdS_4 \times \mathbb{CP}^3$, because supersymmetries do not match [31]. The field theoretical partner of the fundamental string is instead the 1/2 BPS loop discovered in [30] (see [40] for a derivation arising from the low-energy dynamics of heavy W -bosons): the loop couples, in addition to the gauge and scalar fields of the theory, also to the fermions in the bi-fundamental representation of the $U(N) \times U(M)$ gauge group. These ingredients are combined into a superconnection whose holonomy gives the Wilson loop, which can be defined for any representation of the supergroup $U(N|M)$. Supersymmetry is realized through a highly sophisticated mechanism, as a super-gauge transformation, requiring therefore the full non-linear structure of the path-ordering. Actually both loops turn out to be in the same cohomology class, differing by a BRST exact term with respect the localization complex: their quantum expectation value should be therefore the same [30]. The above equivalence has not been checked at weak-coupling, where perturbative computations have been performed just for the 1/6 BPS circle [31–33], the presence of fermions complicates the calculations and rises delicate issues on the regularization procedure. Crucially there are also no examples of loops with fewer supersymmetries, including the known BPS lines and circle as particular cases: it would be interesting to find configurations of this type that could also help to understand better the mysterious cohomological equivalence.

This paper represents a first step towards a systematic study of generalized cusps in ABJ(M) theories: similar configurations have been discussed, at strong coupling, in [34]. We hope that our investigations could stimulate the application of integrability and cohomological techniques in the exact evaluation of non-BPS observables, such as the heavy-bosons static potential. Our main concern here is the construction of a generalized cusp using two 1/2 BPS rays, the study of its supersymmetric properties and its quantum behavior at weak-coupling. The additional R -symmetry deformation is obtained by preserving different $SU(3)$ subgroups on the two lines: from the bosonic point of view this amounts to deform the mass-matrix M_J^I , by rotating two directions of opposite eigenvalues. The fermionic couplings experience a similar deformation and are also explicitly affected by the geometric parameter φ , because they transform as spinors under spatial rotations. We

study the supersymmetry shared by the two rays and we discover that for $\theta = \pm\varphi$ two charges are still globally preserved: in this case the super-gauge transformations, encoding the supersymmetry variation on the two edges of the cusp, become smoothly connected at the meeting point. A key observation made in the original paper of Drukker and Trancanelli [30] was that, for the 1/2 BPS circle, only the trace of the super-holonomy turns out to be supersymmetric invariant and not its super-trace. Conversely the fermionic couplings were assumed to be anti-periodic on the loop: here we examine the same problem on the new BPS configurations. By performing an explicit conformal transformation, we map our cusp on S^2 , obtaining a wedge: the fermionic couplings, constant on the plane, become space-dependent as an effect of the conformal map, and connected by a non-trivial rotation on the upper point of the wedge. In the BPS case this matrix simply appears as an anti-periodicity, and therefore it is again the trace that leads to supersymmetric invariance. The loops constructed in this way are a sort of ABJ(M) version of the DGRT wedge [20] and preserve 1/6 of the original supersymmetries. We consistently define our generalized cusp as the trace of the super-holonomy and attempt the computation of its quantum expectation value in perturbation theory. We observe two basic differences with the analogous four-dimensional computation: first of all the effective propagators here, attaching on one side of the cusp only, are not automatically vanishing and the fermionic sector gives a divergent contribution at one-loop, that has to be regularized and renormalized. Secondly, because of the presence of a supergroup structure, involving fermions coupled to the external lines, it is not obvious to extend the non-abelian exponentiation theorem to this setting: we could not rely on such powerful device to reduce the amount of computations and to properly isolate the cusp anomalous dimensions. Concerning this second point we make the natural assumption that our cusped loops undergo through a "double-exponentiation"

$$\mathcal{W} = \frac{M \exp(V_N) + N \exp(V_M)}{N + M}, \quad (1.4)$$

where the generalized potentials¹ V_N and V_M are simply related by exchanging N with M . A highly non-trivial check of the above assumption is the actual exponentiation of the one-loop term, constraining in particular the structure of the double-poles (in the dimensional regularization parameter $\epsilon = (3 - D)/2$) appearing at two-loops. We find a perfect agreement of our results with the double-exponentiation hypothesis, recovering at the second order in perturbation theory the quadratic contributions coming from the first order one. Our final expression for the unrenormalized V_N is

$$V_N = \left(\frac{2\pi}{\kappa}\right) N \left(\frac{\Gamma(\frac{1}{2} - \epsilon)}{4\pi^{3/2-\epsilon}}\right) (\mu L)^{2\epsilon} \left[\frac{1}{\epsilon} \left(\frac{\cos \frac{\theta}{2}}{\cos \frac{\varphi}{2}} - 2 \right) - 2 \frac{\cos \frac{\theta}{2}}{\cos \frac{\varphi}{2}} \log \left(\sec \left(\frac{\varphi}{2} \right) + 1 \right) \right] + \\ \left(\frac{2\pi}{\kappa}\right)^2 N^2 \left(\frac{\Gamma(\frac{1}{2} - \epsilon)}{4\pi^{3/2-\epsilon}}\right)^2 (\mu L)^{4\epsilon} \left[\frac{1}{\epsilon} \log \left(\cos \frac{\varphi}{2} \right)^2 \left(\frac{\cos \frac{\theta}{2}}{\cos \frac{\varphi}{2}} - 1 \right) + O(1) \right]. \quad (1.5)$$

¹With an abuse of language we have referred to V_N and V_M as the *generalised potentials*. Actually only the coefficient of the $1/\epsilon$ pole has this meaning.

Here k is the Chern-Simons level, L is the length of the lines and μ is the renormalization scale introduced by dimensional regularization. To extract the cusp anomalous dimension we have to carefully subtract the divergences coming from single-leg diagrams: in four dimensions these are usually associated to a linear divergence proportional to the perimeter loop [16]. In three dimensions the situation is quite different and the superconformal case presents further peculiarities, due to the fermionic couplings: we will discuss the subtraction procedure, hoping to clarify the structure of the divergences for these family of loops. The final receipt amounts to subtract, in our second order computation, the one-loop poles associated to single line diagrams, normalizing in this way the final result to the straight line, 1/2 BPS contour

$$V_N^{\text{Ren.}} = \left(\frac{2\pi}{\kappa}\right) N \left(\frac{\Gamma(\frac{1}{2} - \epsilon)}{4\pi^{3/2-\epsilon}}\right) (\mu L)^{2\epsilon} \left[\frac{1}{\epsilon} \left(\frac{\cos \frac{\theta}{2}}{\cos \frac{\varphi}{2}} - 1 \right) - 2 \frac{\cos \frac{\theta}{2}}{\cos \frac{\varphi}{2}} \log \left(\sec \left(\frac{\varphi}{2} \right) + 1 \right) + \log 4 \right] + \\ + \left(\frac{2\pi}{\kappa}\right)^2 N^2 \left(\frac{\Gamma(\frac{1}{2} - \epsilon)}{4\pi^{3/2-\epsilon}}\right)^2 (\mu L)^{4\epsilon} \left[\frac{1}{\epsilon} \log \left(\cos \frac{\varphi}{2} \right)^2 \left(\frac{\cos \frac{\theta}{2}}{\cos \frac{\varphi}{2}} - 1 \right) + O(1) \right]. \quad (1.6)$$

From the above expression we can easily recover the quark-antiquark potential², taking the limit $\varphi \rightarrow \pi$ and following the prescription of [1]

$$V_N^{(s)}(R) = \frac{N}{k} \frac{1}{R} - \left(\frac{N}{k}\right)^2 \frac{1}{R} \log \left(\frac{T}{R} \right). \quad (1.7)$$

We find a logarithmic, non-analytic term in T/R at the second non-trivial order that, as in four dimensions, is expected to disappear when resummation of the perturbative series is performed. In the opposite limit, for large imaginary φ , we get the universal cusp anomaly (using the four-dimensional definition)

$$\gamma_{\text{cusp}} = \frac{N^2}{k^2}, \quad (1.8)$$

reproducing the result obtained directly from the light-like cusp [26].

The plan of the paper is the following: in Section 2 we review the construction of 1/2 BPS Wilson lines in ABJ models, giving us the possibility to introduce the peculiar structures entailed by maximal supersymmetric loops in $\mathcal{N} = 6$ super Chern-Simons-matter theories. Section 3 is devoted to the explicit realization of the generalized cusp: we obtain the appropriate bosonic and fermionic couplings and their deformations and discuss how the supersymmetry properties depends on the relevant parameters. The conformal transformation, mapping the cusp on a wedge of S^2 , is also presented: the periodicity properties of the fermions are derived and BPS observables are obtained taking the trace of the superholonomy. In Section 4 we start the quantum investigation computing the expectation value at the first order in perturbation theory. The two-loop analysis is contained in Section 5. The final result, obtained by summing up all the contributions and performing the

²Actually we have two potentials, $V_N^{(s)}$ and $V_M^{(s)}$, associated respectively to singlets in the $N \times \bar{N}$ and $M \times \bar{M}$ direct product.

renormalization procedure is presented in Section 6, where the peculiar divergences structure of these observables is carefully discussed. Some conclusions and outlooks appear in Section 7. We complete the paper with some appendices, containing our conventions and the technical details of the computations.

2 1/2 BPS straight-line in ABJ theories

We start by reviewing the construction of the 1/2–BPS Wilson line given in [30, 40]: the mechanism leading to its gauge invariance is carefully reconsidered, since it is substantially different from the four dimensional analogue.

The central idea of [30] is to replace the obvious $U(N) \times U(M)$ gauge connection with a super-connection

$$\mathcal{L}(\tau) \equiv -i \begin{pmatrix} i\mathcal{A} & \sqrt{\frac{2\pi}{k}}|\dot{x}|\eta_I\bar{\psi}^I \\ \sqrt{\frac{2\pi}{k}}|\dot{x}|\psi_I\bar{\eta}^I & i\hat{\mathcal{A}} \end{pmatrix} \quad \text{with} \quad \begin{cases} \mathcal{A} \equiv A_\mu \dot{x}^\mu - \frac{2\pi i}{k}|\dot{x}|M_J{}^I C_I \bar{C}^J \\ \hat{\mathcal{A}} \equiv \hat{A}_\mu \dot{x}^\mu - \frac{2\pi i}{k}|\dot{x}|\hat{M}_J{}^I \bar{C}^J C_I, \end{cases} \quad (2.1)$$

belonging to the super-algebra³ of $U(N|M)$. In (2.1) the coordinates $x^\mu(\tau)$ draw the contour along which the loop operator is defined, while $M_J{}^I$, $\hat{M}_J{}^I$, η_I^α and $\bar{\eta}_\alpha^I$ are free parameters. The latter two, in particular, are Grassmann even quantities even though they transform in the spinor representation of the Lorentz group.

The dependence of $\mathcal{L}(\tau)$ on the fields is largely dictated by dimensional analysis and transformation properties. Since the *classical* dimension of the scalars in $D = 3$ is 1/2, they could only appear as bilinears, transforming in the adjoint and thus entering in the diagonal blocks together with the gauge fields. Instead the fermions have dimension 1 and should appear linearly. Since they transform in the bi-fundamentals of the gauge group, they are naturally placed in the off-diagonal entries of the matrix (2.1).

When the contour $x^\mu(\tau)$ is a straight-line S , the invariance under translations along the direction defined by S ensures that all the couplings can be chosen to be independent of τ , *i.e.* constant. Moreover the requirement of having an unbroken $SU(3)$ R –symmetry, as that of the dual string configuration, restricts the couplings $M_J{}^I$, $\hat{M}_J{}^I$, η_I^α and $\bar{\eta}_\alpha^I$ to be of the form

$$\eta_I^\alpha = n_I \eta^\alpha, \quad \bar{\eta}_\alpha^I = \bar{n}^I \bar{\eta}_\alpha, \quad M_J{}^I = p_1 \delta_J^I - 2p_2 n_J \bar{n}^I, \quad \widehat{M}_J{}^I = q_1 \delta_J^I - 2q_2 n_J \bar{n}^I. \quad (2.2)$$

Here n_I and \bar{n}^I are two complex conjugated vectors which transform in the fundamental and anti-fundamental representation and determine the embedding of the $SU(3)$ subgroup in $SU(4)$ ⁴. By rescaling η^α and $\bar{\eta}_\alpha$, we can always choose $n_I \bar{n}^I = 1$. The parameters p_i and q_i in the definition of M and \hat{M} instead control the eigenvalues of the two matrices.

³In Minkowski space-time, where ψ and $\bar{\psi}$ are related by complex conjugation, $\mathcal{L}(\tau)$ belongs to $\mathfrak{u}(N|M)$ if $\bar{\eta} = i(\eta)^\dagger$. In Euclidean space, where the reality condition among spinors are lost, we shall deal with the complexification of this group $\mathfrak{sl}(N|M)$.

⁴In the internal R –symmetry space n_I identifies the direction preserved by the action of the $SU(3)$ subgroup

The free parameters appearing in (2.2) can be then constrained by imposing that the resulting Wilson loop is globally supersymmetric. This issue is subtle: the usual requirement $\delta_{\text{susy}}\mathcal{L}(\tau) = 0$ does not yield any 1/2 BPS solution indeed. We just obtain loop operators which are merely bosonic ($\eta = \bar{\eta} = 0$) and at most 1/6 BPS [31, 33]. In order to obtain 1/2 BPS solution, we must replace $\delta_{\text{susy}}\mathcal{L}(\tau) = 0$ with the weaker condition [30, 40]

$$\delta_{\text{susy}}\mathcal{L}(\tau) = \mathfrak{D}_\tau G \equiv \partial_\tau G + i\{\mathcal{L}, G\}, \quad (2.3)$$

where the r.h.s. is the super-covariant derivative constructed out of the connection $\mathcal{L}(\tau)$ acting on a super-matrix G in $\mathfrak{u}(N|M)$. The requirement (2.3) assures that the action of the relevant supersymmetry charges translates into an infinitesimal super-*gauge* transformation for $\mathcal{L}(\tau)$ and thus the “*traced*” loop operator is invariant.

Now we shall recapitulate the analysis of [30] leading to fix the free parameters in (2.2). However, for future convenience, we shall present the result in a *covariant* notation *i.e.* without referring to a specific form of the straight line.

We start by considering the structure of the infinitesimal gauge parameter in (2.3). Since the supersymmetry transformation of the bosonic fields does not contain any derivative of the fields, the super-matrix G in (2.3) must be anti-diagonal

$$G = \begin{pmatrix} 0 & g_1 \\ \bar{g}_2 & 0 \end{pmatrix} \Rightarrow \mathfrak{D}_\tau G = \begin{pmatrix} \sqrt{\frac{2\pi}{k}}|\dot{x}|(\eta_I \bar{\psi}^I \bar{g}_2 + g_1 \psi_I \bar{\eta}^I) & \mathcal{D}_\tau g_1 \\ \mathcal{D}_\tau \bar{g}_2 & \sqrt{\frac{2\pi}{k}}|\dot{x}|(\bar{g}_2 \eta_I \bar{\psi}^I + \psi_I \bar{\eta}^I g_1) \end{pmatrix}. \quad (2.4)$$

Here \mathcal{D}_τ is the covariant derivative constructed out of the *dressed* bosonic connections \mathcal{A} and $\hat{\mathcal{A}}$ and given by

$$\mathcal{D}_\tau g_1 = \partial_\tau g_1 + i(\mathcal{A} g_1 - g_1 \hat{\mathcal{A}}), \quad \mathcal{D}_\tau \bar{g}_2 = \partial_\tau \bar{g}_2 - i(\bar{g}_2 \mathcal{A} - \hat{\mathcal{A}} \bar{g}_2). \quad (2.5)$$

The condition (2.3) for the anti-diagonal entries first constrains the form of the spinor η and $\bar{\eta}$ to obey the two conditions

$$(\dot{x}^\mu \gamma_\mu)_\alpha{}^\beta = \frac{1}{(\eta\bar{\eta})}|\dot{x}|(\eta^\beta \bar{\eta}_\alpha + \eta_\alpha \bar{\eta}^\beta) \quad (\eta^\beta \bar{\eta}_\alpha - \eta_\alpha \bar{\eta}^\beta) = (\eta\bar{\eta})\delta_\alpha^\beta, \quad (2.6)$$

which assure that the covariant derivatives appearing in the supersymmetry transformations of $\psi\bar{\eta}$ and $\eta\bar{\psi}$ are only evaluated along the circuit. The value of the parameters p_i and q_i , appearing in the matrices M and \hat{M} , is equal to 1 for the same reason.

The requirement (2.3) for the diagonal entries does not yield, instead, new conditions, simply fixing the normalization

$$\eta\bar{\eta} = 2i. \quad (2.7)$$

In particular the vector n_I continues to be unconstrained.

The origin of the superconnection was also investigated from the point of view of the low-energy dynamics of heavy “*W-bosons*” in [40]. It was shown that when the theory is higgsed preserving half of the total supersymmetry, the corresponding low-energy Lagrangian enjoys

a larger gauge invariance, given by the supergroup $U(N|M)$. The light fermions do not decouple from the dynamics, at variance with the case of $\mathcal{N} = 4$ SYM, and their interactions with heavy W -bosons are described by $\eta_I^\alpha, \bar{\eta}_\alpha^I$. The role of the mass-matrix is instead played by M_J^I, \widehat{M}_J^I . This result unveils the physical nature of the potential related to the rectangular Wilson loops, constructed with 1/2 BPS lines in ABJ(M) theories.

Armed with the explicit form for the couplings, we can find twelve supercharges [30] whose action on $\mathcal{L}(\tau)$ can be cast into the form (2.3). There are six supercharges of the Poincaré type⁵,

$$\bar{\theta}^{IJ\beta} = (\bar{\eta}^{I\beta} \bar{v}^J - \bar{\eta}^{J\beta} \bar{v}^I) - i\epsilon^{IJKL} \eta_K^\beta u_L = (\bar{n}^I \bar{v}^J - \bar{n}^J \bar{v}^I) \bar{\eta}^\beta - i\epsilon^{IJKL} n_K u_L \eta^\beta, \quad (2.8)$$

parametrized by two vectors u^I and \bar{v}^I that satisfy $(n_I \bar{v}^I) = (\bar{n}^I u_I) = 0$. We remark that these vectors are really independent in Euclidean signature, while $\bar{v}^I = u_I^\dagger$ as a result of the reality conditions present in the Minkowski case. Next to the above $\bar{\theta}^{IJ}$ we can also identify six super-conformal charges⁶ $\bar{\vartheta}^{IJ\beta}$, whose structure is again given by an expansion of the form (2.8). The origin of this second set of supersymmetries is easily understood: they are obtained by combining the Poincaré supercharges (2.8) with a special conformal transformation in the direction associated to the straight-line.

The analysis presented in [30, 40] also provides the explicit form of the gauge function in terms of the scalar fields, the spinor couplings and the supersymmetry parameters $\bar{\theta}^{IJ}$. In our notation, they take the form

$$g_1 = 2\sqrt{\frac{2\pi}{k}} n_K (\eta \bar{\theta}^{KL}) C_L \quad \text{and} \quad \bar{g}_2 = -\sqrt{\frac{2\pi}{k}} \varepsilon_{IJKL} \bar{n}^K (\bar{\theta}^{IJ} \bar{\eta}) \bar{C}^L. \quad (2.9)$$

Now we come back to analyze the issue of supersymmetry invariance for a generic Wilson loop defined by (2.1) when its variation can be cast into the form (2.3). The finite transformation of the untraced operator

$$W(\tau_1, \tau_0) = \text{P exp} \left(-i \int_{\tau_0}^{\tau_1} d\tau \mathcal{L}(\tau) \right), \quad (2.10)$$

under the gauge transformation generated by $G(\tau)$ in (2.4), can be written as [40]

$$W(\tau_1, \tau_0) \mapsto U^{-1}(\tau_1) W(\tau_1, \tau_0) U(\tau_0), \quad (2.11)$$

where $U(\tau) = e^{iG(\tau)}$. For a closed path γ [$\gamma(\tau_0) = \gamma(\tau_1)$], we must carefully consider the boundary conditions obeyed by the gauge functions g_1 and \bar{g}_2 in order to define the gauge invariant operator. If they are *periodic*, i.e. $g_1(\tau_0) = g_1(\tau_1)$ and $\bar{g}_2(\tau_0) = \bar{g}_2(\tau_1)$, we

⁵Recall that the counting is performed in terms of complex supercharges in Euclidean space-time, while we use *real* supercharges in Minkowski signature.

⁶We have parametrized a generic supercharge as follows

$$\bar{\Theta}^{IJ} = \bar{\theta}^{IJ} + x^\mu \gamma_\mu \bar{\vartheta}^{IJ},$$

where $\bar{\theta}^{IJ}$ generates the Poincaré supersymmetries, while $\bar{\vartheta}^{IJ}$ yields the conformal ones.

find that $U(\tau_0) = U(\tau_1)$ and a gauge invariant operator is obtained by taking the usual super-trace

$$\mathcal{W} = \text{Str}(W). \quad (2.12)$$

Actually it is the super-trace to be invariant under similitude transformations. However we can have different situations: in [30] it was examined another 1/2-BPS loop, the circle, and pointed out that the function g_1 and \bar{g}_2 are anti-periodic in this case. Consequently the untraced operator, because $U(\tau_1) = U^{-1}(\tau_0)$, transforms as follows

$$W(\tau_1, \tau_0) \mapsto U(\tau_0)W(\tau_1, \tau_0)U(\tau_0). \quad (2.13)$$

To construct a supersymmetric operator, we first observe that

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} U(\tau_0) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = U^{-1}(\tau_0) \quad (2.14)$$

for a gauge transformation generated by the matrix G in (2.4). Then the operator

$$\mathcal{W} = \text{Str} \left[\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \text{Pexp} \left(-i \int_{\tau_0}^{\tau_1} d\tau \mathcal{L}(\tau) \right) \right] \equiv \text{Tr} \left[\text{Pexp} \left(-i \int_{\tau_0}^{\tau_1} d\tau \mathcal{L}(\tau) \right) \right] \quad (2.15)$$

turns out to be invariant. In the case of a straight line the situation is more intricate, since we deal with an open infinite circuit. The invariance under supersymmetry is recovered by choosing a set of suitable boundary conditions for the fields, in particular for the scalars appearing in the definition of g_1 and \bar{g}_2 . The naive statement that they must vanish when $\tau = \pm\infty$ seems to leave open a double possibility for defining a supersymmetric operator

$$\begin{aligned} \mathcal{W}_- &= \frac{1}{N-M} \text{Str} \left[\text{Pexp} \left(-i \int_{\tau_0}^{\tau_1} d\tau \mathcal{L}(\tau) \right) \right] \text{ or} \\ \mathcal{W}_+ &= \frac{1}{N+M} \text{Tr} \left[\text{Pexp} \left(-i \int_{\tau_0}^{\tau_1} d\tau \mathcal{L}(\tau) \right) \right], \end{aligned} \quad (2.16)$$

since $U(\pm\infty) = 1$ and $W(\tau_1, \tau_0)$ itself is invariant. We shall consider in our explicit quantum computation the second possibility: as we will see, the trace is the correct option to generate BPS observables from closed contours, connected to ours through conformal transformations. It also seems to provide a result consistent with the interpretation of the Wilson loop in terms of quark-antiquark potentials.

3 The generalized cusp

We discuss here in detail the Wilson loop observables we will study in the rest of the paper. After constructing the bosonic and fermionic couplings for the generalized cusp, we study the possibility to find novel BPS configurations. We determine the BPS conditions for the cusp parameters and derive the explicit form of the related supercharges. Finally we map our new configurations on the sphere S^2 , by means of conformal transformations, and we obtain a non-trivial BPS deformation of the BPS circle constructed in [30].

3.1 Bosonic and fermionic couplings

We start by considering the theory on the Euclidean space-time. We shall consider two rays in the plane $(1, 2)$ intersecting at the origin as illustrated in fig. 1. The angle between the rays is $\pi - \varphi$, such that for $\varphi = 0$ they form a continuous straight line. The path in fig. 1 is given by

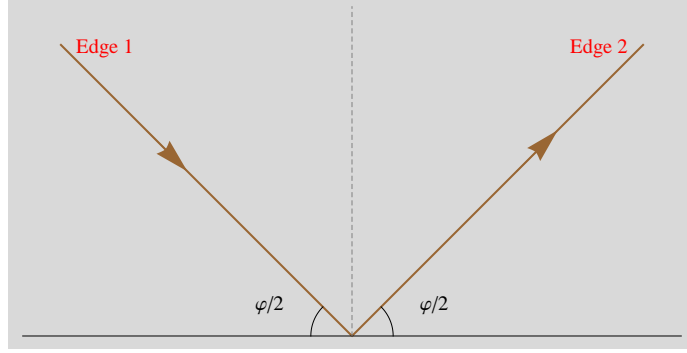


Figure 1. Eq. (3.1) represents a planar cusp, whose angular extension is given by $\pi - \varphi$.

$$x^0 = 0 \quad x^1 = s \cos \frac{\varphi}{2} \quad x^2 = |s| \sin \frac{\varphi}{2} \quad -\infty \leq s \leq \infty. \quad (3.1)$$

The fermionic couplings on each straight-line possess the factorized structure discussed in the previous section, *i.e.*

$$\eta_{iM}^\alpha = n_{iM} \eta^\alpha \quad \text{and} \quad \bar{\eta}_{i\alpha}^M = \bar{n}_i^M \bar{\eta}_{i\alpha}. \quad (3.2)$$

The additional index $i = 1, 2$ in (3.2) specifies which edge of the cusp we are considering. For $i = 1$, $\eta_{1\alpha}$ is constructed as the eigenspinor of eigenvalue 1 of the matrix $\frac{\dot{x}_1^\mu}{|\dot{x}_1|} \gamma_\mu$:

$$\frac{\dot{x}_1^\mu}{|\dot{x}_1|} \gamma_\mu \bar{\eta}_1 = \left(\cos \frac{\varphi}{2} \gamma_1 - \sin \frac{\varphi}{2} \gamma_2 \right) \bar{\eta}_1 = \bar{\eta}_1 \quad \Rightarrow \quad \bar{\eta}_{1\alpha} = i \begin{pmatrix} e^{i\frac{\varphi}{4}} \\ e^{-i\frac{\varphi}{4}} \end{pmatrix} \quad (3.3)$$

as a result⁷ of the two constraints (2.6). Similarly the spinor η_1^α obeys the hermitian conjugate of the above equation and thus $\eta_1^\alpha \propto (\bar{\eta}_{1\alpha})^\dagger$. The condition (2.7) fixes the relative normalization and we find

$$\eta_1^\alpha = (e^{-i\frac{\varphi}{4}} \quad e^{i\frac{\varphi}{4}}). \quad (3.4)$$

On the other hand the R -symmetry part of the couplings is arbitrary and in fact n_{1M} and \bar{n}_1^M are totally unconstrained. For future convenience we choose

$$n_{1M} = (\cos \frac{\theta}{4} \quad \sin \frac{\theta}{4} \quad 0 \quad 0) \quad \text{and} \quad \bar{n}_1^M = \begin{pmatrix} \cos \frac{\theta}{4} \\ \sin \frac{\theta}{4} \\ 0 \\ 0 \end{pmatrix}. \quad (3.5)$$

⁷We have dropped a global phase in $\bar{\eta}_1$ since it can be eliminated through a global $U(1)$ redefinition of the matter fields.

On the second edge, again as a result of (2.6), $\bar{\eta}_2$ must be the eigenspinor of eigenvalue 1 of $\frac{\dot{x}_2^\mu}{|x_2|}\gamma_\mu$ and following the same route we get

$$\bar{\eta}_{2\alpha} = ie^{i\delta} \begin{pmatrix} e^{-i\frac{\varphi}{4}} \\ e^{i\frac{\varphi}{4}} \end{pmatrix} \quad \eta_2^\alpha = e^{-i\delta} (e^{i\frac{\varphi}{4}} \quad e^{-i\frac{\varphi}{4}}). \quad (3.6)$$

The arbitrary phase δ cannot be reabsorbed into a redefinition of the fields without altering the structure of the fermionic couplings on the first edge. For the R -symmetry sector in (3.2) we instead set

$$n_{2M} = (\cos \frac{\theta}{4} \quad -\sin \frac{\theta}{4} \quad 0 \quad 0) \quad \text{and} \quad \bar{n}_2^M = \begin{pmatrix} \cos \frac{\theta}{4} \\ -\sin \frac{\theta}{4} \\ 0 \\ 0 \end{pmatrix}. \quad (3.7)$$

The two matrices which couple the scalars are then determined through the relations (2.2), which give M and \widetilde{M} in terms of n and \bar{n} . On the two edges we have respectively

$$M_{1J}^I = \widehat{M}_{1J}^I = \begin{pmatrix} -\cos \frac{\theta}{2} & -\sin \frac{\theta}{2} & 0 & 0 \\ -\sin \frac{\theta}{2} & \cos \frac{\theta}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad M_{2J}^I = \widehat{M}_{2J}^I = \begin{pmatrix} -\cos \frac{\theta}{2} & \sin \frac{\theta}{2} & 0 & 0 \\ \sin \frac{\theta}{2} & \cos \frac{\theta}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad (3.8)$$

3.2 Intermediate BPS configurations

In the following we would like to explore if there is a choice of the $(\varphi, \theta, \delta)$ such that the generalized cusp turns out to be BPS⁸. These configurations may provide useful checks for the perturbative computations, but they can also provide a tool for addressing the issue of nonperturbative computations [2].

Let us consider one of the Poincaré supersymmetries preserved by the first edge of the cusp in fig. 1. As discussed in sec. 2, it admits the following expansion

$$\theta_1^{IJ\beta} = (\bar{\eta}_1^{I\beta} \bar{v}_1^J - \bar{\eta}_1^{J\beta} \bar{v}_1^I) - i\epsilon^{IJKL} \eta_{1K}^\beta u_{1L} = (\bar{n}_1^I \bar{v}_1^J - \bar{n}_1^J \bar{v}_1^I) \bar{\eta}_1^\beta - i\epsilon^{IJKL} n_{1K} u_{1L} \eta_1^\beta, \quad (3.9)$$

where η_{1K} $\bar{\eta}_1^I$ are the spinor couplings on the first line. The choice of the two vectors u_1 and \bar{v}_1 selects the charge that we are considering. We observe that if (3.9) defines a supercharge shared with the second edge it must admit a similar expansion in terms of the spinor couplings of the second line. Expanding θ_1^{IJ} in the basis provided by η_2 and $\bar{\eta}_2$, we obtain the following system of equation

$$-i\epsilon^{IJKL} n_{2K} u_{2L} = -ie^{i\delta} \epsilon^{IJKL} n_{1K} u_{1L} \cos \frac{\varphi}{2} + (\bar{n}_1^I \bar{v}_1^J - \bar{n}_1^J \bar{v}_1^I) e^{i\delta} \sin \frac{\varphi}{2} \quad (3.10a)$$

$$(\bar{n}_2^I \bar{v}_2^J - \bar{n}_2^J \bar{v}_2^I) = ie^{-i\delta} \epsilon^{IJKL} n_{1K} u_{1L} \sin \frac{\varphi}{2} + (\bar{n}_1^I \bar{v}_1^J - \bar{n}_1^J \bar{v}_1^I) e^{-i\delta} \cos \frac{\varphi}{2}. \quad (3.10b)$$

When this set of equations can be consistently solved both for u_2 and \bar{v}_2 , we have found a candidate BPS configuration. To begin with, we shall multiply (3.10a) by n_{2J} . The

⁸Of course we have an obvious one: $\varphi = \theta = \delta = 0$.

resulting condition does not contain u_2 and \bar{v}_2 : it is actually a constraint on the supercharge $\bar{\theta}_1^{IJ}$

$$\epsilon^{IJKL} n_{2J} n_{1K} u_{1L} \cos \frac{\varphi}{2} + i \left(\bar{n}_1^I (n_{2J} \bar{v}_1^J) - \cos \frac{\theta}{2} \bar{v}_1^I \right) \sin \frac{\varphi}{2} = 0. \quad (3.11)$$

If we project (3.11) onto the direction n_{1I} , we have immediately

$$(n_{2J} \bar{v}_1^J) \sin \frac{\varphi}{2} e^{i\delta} = 0 \quad \Rightarrow \quad (n_{2J} \bar{v}_1^J) = 0. \quad (3.12)$$

and consequently from (3.11)

$$\bar{v}_1^I = -i \frac{\cot \frac{\varphi}{2}}{\cos \frac{\theta}{2}} \epsilon^{IJKL} n_{2J} n_{1K} u_{1L}. \quad (3.13)$$

Next we multiply (3.10b) by $\epsilon_{IJMN} \bar{n}_2^M$. Again the dependence on u_2 and v_2 drops out and we end up with the following constraint

$$0 = i \left(\cos \frac{\theta}{2} u_{1N} - n_{1N} (\bar{n}_1^M u_{1M}) \right) \sin \frac{\varphi}{2} + \epsilon_{IJMN} \bar{n}_1^I \bar{v}_1^J \bar{n}_2^M \cos \frac{\varphi}{2}, \quad (3.14)$$

which is equivalent to

$$u_{1L} = i \frac{\cot \frac{\varphi}{2}}{\cos \frac{\theta}{2}} \epsilon_{RSM L} \bar{n}_1^R \bar{v}_1^S \bar{n}_2^M. \quad (3.15)$$

The relations (3.13) and (3.15) are consistent if and only if

$$\bar{v}_1^I = -i \frac{\cot \frac{\varphi}{2}}{\cos \frac{\theta}{2}} \epsilon^{IJKL} n_{2J} n_{1K} u_{1L} = \frac{\cot^2 \frac{\varphi}{2}}{\cos^2 \frac{\theta}{2}} \epsilon^{IJKL} n_{2J} n_{1K} \epsilon_{RSM L} \bar{n}_1^R \bar{v}_1^S \bar{n}_2^M = \frac{\cot^2 \frac{\varphi}{2}}{\cot^2 \frac{\theta}{2}} v_1^I, \quad (3.16)$$

namely $\cot^2 \frac{\varphi}{2} = \cot^2 \frac{\theta}{2}$. Therefore for $\theta = \pm\varphi$ we expect that the loop operator defined in the previous section is BPS. In fact for this value of the parameters we can find an explicit and simple solution for u_2 and \bar{v}_2

$$u_{2I} = e^{i\delta} u_{1I} \quad \text{and} \quad \bar{v}_2^I = e^{-i\delta} v_1^I. \quad (3.17)$$

We remark we still need another property to fully confirm the presence of a BPS configuration at $\theta = \pm\varphi$: we should prove that the gauge functions g_1 and \bar{g}_2 on the two edges define a globally well-defined gauge transformation, which is continuous when we cross the cusp. The values of g_1 on the two edges are given by

$$[\text{FIRST EDGE}] : 4i \sqrt{\frac{2\pi}{k}} \bar{v}_1^L C_L \quad [\text{SECOND EDGE}] : 4i \sqrt{\frac{2\pi}{k}} \bar{v}_1^L e^{-i\delta} C_L, \quad (3.18)$$

while for \bar{g}_2 we find

$$[\text{FIRST EDGE}] : 4 \sqrt{\frac{2\pi}{k}} u_{1L} \bar{C}^L \quad [\text{SECOND EDGE}] : 4 \sqrt{\frac{2\pi}{k}} e^{i\delta} u_{1L} \bar{C}^L. \quad (3.19)$$

Only for $\delta = 0$ the two gauge function are continuous through the cusp. Summarizing, for the $\theta = \pm\varphi$ and $\delta = 0$ the generalized cusp of fig.1 is BPS. The preserved Poincarè

supercharges in terms of the quantity of the first line can be then written in the following two equivalent ways

$$\begin{aligned}\bar{\theta}_1^{IJ\beta} &= -i \left(\frac{\bar{n}_1^I}{\sin \frac{\varphi}{2}} \epsilon^{JMKL} n_{2M} n_{1K} u_{1L} - \frac{\bar{n}_1^J}{\sin \frac{\varphi}{2}} \epsilon^{IMKL} n_{2M} n_{1K} u_{1L} \right) \bar{\eta}_1^\beta - i \epsilon^{IJKL} n_{1K} u_{1L} \eta_1^\beta = \\ &= (\bar{n}_1^I \bar{v}_1^J - \bar{n}_1^J \bar{v}_1^I) \bar{\eta}_1^\beta - \frac{1}{\sin \frac{\varphi}{2}} \left(\left(\bar{n}_2^I - \cos \frac{\varphi}{2} \bar{n}_1^I \right) \bar{v}_1^I - \left(\bar{n}_2^J - \cos \frac{\varphi}{2} \bar{n}_1^J \right) \bar{v}_1^I \right) \eta_1^\beta.\end{aligned}\quad (3.20)$$

The vector u_1^I in the first line of (3.20) and the vector \bar{v}_1^I in the second one must obey $(n_{2J} \bar{v}_1^J) = (n_{1J} \bar{v}_1^J) = 0$ and $(\bar{n}_{2J} u_1^J) = (\bar{n}_{1J} u_1^J) = 0$ respectively. Thus we have two shared Poincaré supercharges.

A remark on the conformal supercharges $\bar{\vartheta}^{IJ}$ is now in order: for each edge of the cusp they admit the same expansion (2.8) which was obtained for the Poincaré ones. The above analysis implies therefore that there are two shared superconformal charges as well.

3.3 Mapping the cusp to the spherical wedge

Recently [2] it was noticed that the DGRT spherical wedge [20], which is a BPS loop operator, can be used to extract nonperturbative information about the generalized cusp

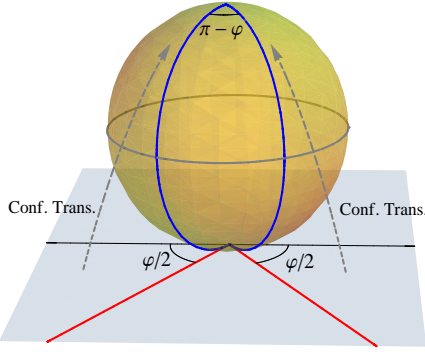


Figure 2. The cusp in the plane $x^0 = 1/2$ is mapped into the spherical wedge under the conformal transformation generated by the vector $(1, 0, 0)$.

investigate what is the image of the operator defined in subsect. 3.1 when we map the plane supporting it onto a sphere S^2 .

Our starting point is not the cusp in the plane $(1, 2)$, as in subsect. 3.1, but for simplicity one located in the plane $x^0 = 1/2$, *i.e.*

$$x^0 = 1/2 \quad x^1 = s \cos \frac{\varphi}{2} \quad x^2 = |s| \sin \frac{\varphi}{2} \quad -\infty \leq s \leq \infty. \quad (3.22)$$

The plane $x^0 = 1/2$ can be mapped into the usual unit sphere centered in the origin through the special conformal transformation generated by the vector $b^\mu = (1, 0, 0)$

$$y^0 = \frac{x^0 - x^\mu x_\mu}{1 - 2x^0 + x^\mu x_\mu}, \quad y^1 = \frac{x^1}{1 - 2x^0 + x^\mu x_\mu}, \quad y^2 = \frac{x^2}{1 - 2x^0 + x^\mu x_\mu}. \quad (3.23)$$

Then the image of the original contour is the wedge illustrated in fig. 2. The new path in terms of s is given by

$$y^0 = \frac{\frac{1}{4} - s^2}{\frac{1}{4} + s^2}, \quad y^1 = \frac{s \cos \frac{\varphi}{2}}{\frac{1}{4} + s^2}, \quad y^2 = \frac{|s| \sin \frac{\varphi}{2}}{\frac{1}{4} + s^2}. \quad (3.24)$$

When s ranges from $-\infty$ to ∞ we move from the south of the sphere $(-1, 0, 0)$ to the north pole $(1, 0, 0)$ [$s = 0$] and back to the south pole. The usual parametrization in polar coordinates is recovered by performing the substitution $s = \frac{1}{2} \tan \frac{t}{2}$ in (3.24)

$$y^0 = \cos t, \quad y^1 = \sin t \cos \frac{\varphi}{2}, \quad y^2 = |\sin t| \sin \frac{\varphi}{2} \quad (3.25)$$

where $t \in [-\pi, \pi]$. The effect of the change of coordinates (3.23) on the fermionic couplings η_I and $\bar{\eta}^I$ is more interesting and straightforward to determine once we recall the result of a finite special conformal transformation on a spinor field⁹ (see *e.g.* [41]). Comparing the anti-diagonal term of the matrix (2.1) in the two different coordinates, we find for instance

$$|\dot{y}| \eta'_I \bar{\psi}^I(y) = |\dot{x}| (1 - 2(b \cdot x) + b^2 x^2)^{-1/2} \eta'_I (\mathbb{1} - b^\mu \gamma_\mu x^\alpha \gamma_\alpha) \bar{\psi}^I(x) = |\dot{x}| \eta_I \bar{\psi}^I(x), \quad (3.26)$$

namely

$$\eta'_I = \eta_I \frac{(\mathbb{1} - x^\alpha \gamma_\alpha b^\mu \gamma_\mu)}{\sqrt{1 - 2(b \cdot x) + b^2 x^2}}. \quad (3.27)$$

In other words, in the case of the spinor couplings, the effect of mapping the cusp into spherical wedge translates into a local rotation defined by the matrix appearing in (3.27). We have a different rotation on each edge

EDGE I: ($s < 0$)

$$R_I = \begin{pmatrix} \frac{1}{\sqrt{4s^2+1}} & \frac{2e^{-\frac{i\varphi}{2}} s}{\sqrt{4s^2+1}} \\ -\frac{2e^{-\frac{i\varphi}{2}} s}{\sqrt{4s^2+1}} & \frac{1}{\sqrt{4s^2+1}} \end{pmatrix} = \begin{pmatrix} \cos \frac{t}{2} & e^{-\frac{i\varphi}{2}} \sin \frac{t}{2} \\ -e^{-\frac{i\varphi}{2}} \sin \frac{t}{2} & \cos \frac{t}{2} \end{pmatrix} \quad (3.28a)$$

EDGE II: ($s > 0$)

$$R_{II} = \begin{pmatrix} \frac{1}{\sqrt{4s^2+1}} & \frac{2e^{-\frac{i\varphi}{2}} s}{\sqrt{4s^2+1}} \\ -\frac{2e^{-\frac{i\varphi}{2}} s}{\sqrt{4s^2+1}} & \frac{1}{\sqrt{4s^2+1}} \end{pmatrix} = \begin{pmatrix} \cos \frac{t}{2} & e^{-\frac{i\varphi}{2}} \sin \frac{t}{2} \\ -e^{-\frac{i\varphi}{2}} \sin \frac{t}{2} & \cos \frac{t}{2} \end{pmatrix}. \quad (3.28b)$$

We have expressed the rotation matrices in terms of both the original parameter s and the parameter $t = 2 \arctan(2s)$. Now the fermionic couplings on the two sides of the wedge are obtained by rotating the old ones by means of the two matrices R_I and R_{II}

$$v_1^\alpha = (\eta_1 R_I)^\alpha = (e^{-i\frac{\varphi}{4}} (\cos \frac{t}{2} - \sin \frac{t}{2}) \quad e^{i\frac{\varphi}{4}} (\cos \frac{t}{2} + \sin \frac{t}{2})) \quad (3.29)$$

⁹In three dimensions, the finite form of a special conformal transformation on a spinor field is

$$\psi'(y) = (1 - 2(b \cdot x) + b^2 x^2)^{1/2} (\mathbb{1} - b^\mu \gamma_\mu x^\alpha \gamma_\alpha) \psi(x).$$

$$v_2^\alpha = (\eta_2 R_{II})^\alpha = \begin{pmatrix} e^{i\frac{\varphi}{4}} (\cos \frac{t}{2} - \sin \frac{t}{2}) & e^{-i\frac{\varphi}{4}} (\cos \frac{t}{2} + \sin \frac{t}{2}) \end{pmatrix} \quad (3.30)$$

and obviously $\bar{v}_{1,2\alpha} = i(v_{1,2}^\alpha)^\dagger$. The matrices M and \widehat{M} which couple the scalars to the loop are instead unaffected by the special conformal transformation.

Next we consider the effect of the change of variables (3.23) on the preserved super-charges of subsect. 3.2. The super-conformal Killing spinors $(\bar{\theta}^{IJ}, \bar{\vartheta}^{IJ})$ of the cusp are transformed into

$$(\bar{\theta}^{IJ}, \bar{\vartheta}^{IJ}) \mapsto (\bar{\theta}^{IJ}, \bar{\vartheta}'^{IJ}) = (\bar{\theta}^{IJ}, \bar{\vartheta}^{IJ} + \bar{\theta}^{IJ}(b^\mu \gamma_\mu)) = (\bar{\theta}^{IJ}, \bar{\vartheta}^{IJ} + \bar{\theta}^{IJ} \gamma_0). \quad (3.31)$$

The loop operator defined by this spherical wedge is preserved by the conformal Killing spinors with a structure given by $\bar{\Theta}^{IJ} = \bar{\theta}^{IJ}(1 + y^\mu \gamma_0 \gamma_\mu) + y^\mu \bar{\vartheta}^{IJ} \gamma_\mu$.

In doing the conformal transformation we have effectively compactified the contour and we have to understand what happens to the gauge functions at the south pole: continuity of the gauge transformations at north pole is instead inherited by the BPS properties of the open cusp. It is a straightforward exercise to compute the spinor contractions which are relevant in determining the gauge function g_1 and \bar{g}_2 at the points $t = -\pi$ and $t = \pi$:

$$\begin{aligned} g_1|_{t=-\pi} &= 4i\sqrt{\frac{2\pi}{k}}(\bar{v}_1^L C_L) & g_1|_{t=\pi} &= -4i\sqrt{\frac{2\pi}{k}}(\bar{v}_2^L C_L) \\ \bar{g}_2|_{t=-\pi} &= 4\sqrt{\frac{2\pi}{k}}(\tilde{u}_{1L} \bar{C}^L) & \bar{g}_2|_{t=\pi} &= -4\sqrt{\frac{2\pi}{k}}(\tilde{u}_{2L} \bar{C}^L) \end{aligned} \quad (3.32)$$

where we used that ϑ^{IJ} admits the same expansion of the Poincaré supercharges in terms of two vectors \tilde{u}_i and \tilde{v}_i . These last two vectors will obey the same constraint of u_i and \bar{v}_i and in particular for the shared supercharges $\bar{v}_1^I = \bar{v}_2^I$ and $\tilde{u}_1^I = \tilde{u}_2^I$. We see the gauge functions (3.32) are anti-periodic and consequently, to have a BPS loop, we have to take the trace to obtain a *supersymmetric* wedge on S^2 . This is consistent with the result of [30], our wedge being a non-trivial BPS deformation of the BPS circle. It is interesting that within our construction the antiperiodicity of gauge functions appears as an effect of the conformal mapping, rather than being assumed from the beginning. The presence of such supersymmetric configurations suggests also that it should be possible to construct a general class of BPS loops on S^2 , representing the ABJ analogue of the DGRT loops of $\mathcal{N} = 4$. The explicit construction and the quantum analysis of this new family, as well as of the analogue of Zarembo's loops in superconformal Chern-Simons theories, will be the subjects of a separate publication [42].

4 Quantum results

We shall compute the expectation value of the generalized *cusp* operator up to the second order in the coupling constant $(\frac{2\pi}{\kappa})$. So far there are very few results about the perturbative properties of supersymmetric Wilson loops in ABJ(M) theories and they are all strictly confined to the 1/6 BPS *bosonic* case [31–33]. We remark that even the matrix model [30]

- believed to capture the exact result for the 1/2 BPS circle - has not been verified by explicit Feynman diagrams computations.

The quantum holonomy of the super-connection \mathcal{L} in a representation \mathcal{R} of the supergroup $U(N|M)$ is by definition

$$\langle \mathcal{W}_{\mathcal{R}} \rangle = \frac{1}{\dim \mathcal{R}} \int \mathcal{D}[A, \hat{A}, C, \bar{C}, \psi, \bar{\psi}] e^{-S_{ABJ}} \text{Tr}_{\mathcal{R}} \left[\text{P exp} \left(-i \int_{\Gamma} d\tau \mathcal{L}(\tau) \right) \right], \quad (4.1)$$

where S_{ABJ} is the Euclidean action for $ABJ(M)$ theories (the part relevant for our computation is spelled out in app. A). In the following \mathcal{R} will be chosen to be the fundamental representation.

In order to evaluate $\langle \mathcal{W}_{\mathcal{R}} \rangle$ we shall first focus our attention on the upper left $N \times N$ block of the super-matrix appearing in (4.1). For this sub-sector the trace is obviously taken in the fundamental representation \mathbf{N} of the first gauge group. The result for the lower diagonal block can be then recovered from this analysis by exchanging N with M . Our perturbative computation requires to expand the path-exponential in (4.1) up to the fourth order. The terms in this expansion relevant for the upper block include both *bosonic* and *fermionic* monomials:

$$\begin{aligned} \mathbb{W}_{\mathbf{N}} = \text{Tr}_{\mathbf{N}} & \left[1 + i \int_{\Gamma} d\tau_1 \mathcal{A}_1 - \int_{\Gamma} d\tau_{1>2} \left(\mathcal{A}_1 \mathcal{A}_2 - (\eta \bar{\psi})_1 (\psi \bar{\eta})_2 \right) \right. \\ & - i \int_{\Gamma} d\tau_{1>2>3} \left(\mathcal{A}_1 \mathcal{A}_2 \mathcal{A}_3 + \frac{2\pi}{k} [(\eta \bar{\psi})_1 (\psi \bar{\eta})_2 \mathcal{A}_3 + (\eta \bar{\psi})_1 \hat{\mathcal{A}}_2 (\psi \bar{\eta})_3 + \mathcal{A}_1 (\eta \bar{\psi})_2 (\psi \bar{\eta})_3] \right) \\ & + \int_{\Gamma} d\tau_{1>2>3>4} \left(\left(\frac{2\pi}{\kappa} \right)^2 (\eta \bar{\psi})_1 (\psi \bar{\eta})_2 (\eta \bar{\psi})_3 (\psi \bar{\eta})_4 + \mathcal{A}_1 \mathcal{A}_2 \mathcal{A}_3 \mathcal{A}_4 - \right. \\ & - \left(\frac{2\pi}{\kappa} \right) \mathcal{A}_1 \mathcal{A}_2 (\eta \bar{\psi})_3 (\psi \bar{\eta})_4 - \left(\frac{2\pi}{\kappa} \right) \mathcal{A}_1 (\eta \bar{\psi})_2 \hat{\mathcal{A}}_3 (\psi \bar{\eta})_4 - \left(\frac{2\pi}{\kappa} \right) (\eta \bar{\psi})_1 \hat{\mathcal{A}}_2 \hat{\mathcal{A}}_3 (\psi \bar{\eta})_4 - \\ & \left. \left. - \left(\frac{2\pi}{\kappa} \right) \mathcal{A}_1 (\eta \bar{\psi})_2 (\psi \bar{\eta})_3 \mathcal{A}_4 - \left(\frac{2\pi}{\kappa} \right) (\eta \bar{\psi})_1 \hat{\mathcal{A}}_2 (\psi \bar{\eta})_3 \mathcal{A}_4 - \left(\frac{2\pi}{\kappa} \right) (\eta \bar{\psi})_1 (\psi \bar{\eta})_2 \mathcal{A}_3 \mathcal{A}_4 \right) \right]. \end{aligned} \quad (4.2)$$

In (4.2) we have introduced a shorthand notation for the circuit parameter dependence of the fields, namely $\mathcal{A}_i = \mathcal{A}(x_i)$ with $x_i = x(\tau_i)$. We have also suppressed the spinor and $SU(4)_R$ indices and choosen the parametrization with $|\dot{x}| = 1$. The expression above is not symmetric in the exchange of the two gauge groups: the symmetry between them will be recovered when considering also the contribution coming from the lower right $M \times M$ block.

4.1 One-loop analysis

The first non-trivial contributions are proportional to $\left(\frac{2\pi}{\kappa} \right)$ and involve both bosonic and fermionic diagrams. They are listed in fig. 3. Differently from what occurs for the $\mathcal{N} = 4$ generalized cusp, the diagrams which involve only one edge do not vanish when we add

them up. The situation is a little more intricate and it is actually convenient to deal with them separately, also in view of the two-loop computation.

The evaluation of the diagrams in fig. 3 obviously encounter UV divergences which originate from the part of the integration region where the propagator endpoints coincide. To tame these divergences we will extensively use dimensional regularization. However regularizing Chern-Simons-matter theories going off-dimensions raises some concerns because of the presence of the anti-symmetric $\epsilon^{\mu\nu\rho}$ tensor. We will follow the DRED scheme, shifting the dimension to $d = 3 - 2\epsilon$ while keeping the Dirac algebra and $\epsilon^{\mu\nu\rho}$ tensor strictly in 3 dimensions. Note that this breaks the conformal invariance introducing a mass scale $\mu^{2\epsilon}$ that keeps the action dimensionless. We will also need an explicit IR regulator L , representing the finite length of the two rays forming the cusp: because of the underlying conformal invariance we expect that it could be always scaled away, combining into the $(\mu L)^{2\epsilon}$ to some powers that weights the relevant Feynman integrals.

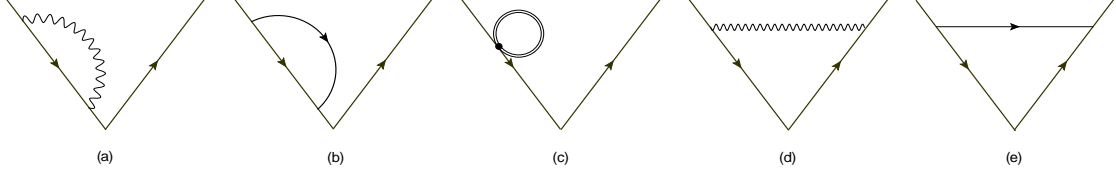


Figure 3. At one-loop order there are only two classes of diagrams: single-edge diagrams [(a), (b) and (c)] and exchange diagrams [(d) and (e)]. The scalar field enters the loop operator only through the composite bilinear $M_J^I C_I \bar{C}^J$, and its conjugate $\bar{M}_J^I \bar{C}^J C_I$, hence the exchange of a single scalar is not permitted.

We start by considering the bosonic diagrams: the scalar tadpole of fig. 3.(c), that originates from the first term in the expansion (4.2), vanishes in our regularization procedure and we can safely forget its existence in the following. The bosonic contributions in fig. 3.(a) and 3.(d) stems from the term with two gauge fields in (4.2) and they can be cast into a single path-ordered integral,

$$\mathfrak{B}^{(1)} = iN^2 \left(\frac{\mu^{2\epsilon}}{\kappa} \right) \frac{\Gamma(\frac{3}{2} - \epsilon)}{\pi^{\frac{1}{2} - \epsilon}} \int_{\Gamma} d\tau_{1>2} \frac{\epsilon_{\mu\nu\rho} \dot{x}_1^\mu \dot{x}_2^\nu (\dot{x}_1 - \dot{x}_2)^\rho}{|x_1 - x_2|^{3-2\epsilon}} = 0, \quad (4.3)$$

whose integrand vanishes for any planar loop due to the antisymmetry of the ϵ -tensor¹⁰. We have used here the explicit form of the Chern-Simons propagator in position space, presented in App. B. We remark that the same result would have been obtained if we have used 1/6 BPS lines, in spite of the different structure of the mass-matrix M_J^I .

Next we discuss the fermionic diagrams in fig. 3. They represent the true novelty of the present calculation and originate from taking the vacuum expectation value of the fermionic bilinear in the first line of (4.2)

$$\mathfrak{F}^{(1)} = \left(\frac{2\pi}{\kappa} \right) \mu^{2\epsilon} \int_{\tau_2 < \tau_1} d\tau_1 d\tau_2 \langle \text{Tr}_{\mathbf{N}} [(\eta\bar{\psi})_1 (\psi\bar{\eta})_2] \rangle = \left(\frac{2\pi}{\kappa} \right) \mu^{2\epsilon} (\mathcal{F}_b + \mathcal{F}_e), \quad (4.4)$$

¹⁰In general one should also take into account the possibility to have framing contribution [43]. We assume here that our computation can be consistently done at zero framing.

where we have denoted with \mathcal{F}_b and \mathcal{F}_e the contributions corresponding to the graphs 3.(b) and 3.(e) respectively. At the lowest order, the vacuum expectation value in (4.4) is simply obtained by contracting the fermion propagator (B.4) with the spinors η and $\bar{\eta}$. We find

$$\langle \text{Tr}_{\mathbf{N}} [(\eta\bar{\psi})_1(\psi\bar{\eta})_2] \rangle = iMN \frac{\Gamma(1/2 - \epsilon)}{4\pi^{3/2-\epsilon}} \eta_{L1} \gamma^\mu \bar{\eta}_2^L \partial_{x_1^\mu} \left(\frac{1}{(x_{12}^2)^{1/2-\epsilon}} \right), \quad (4.5)$$

where $x_{ij} = x_i - x_j$. The fermion bilinear $\eta_{L1} \gamma^\mu \bar{\eta}_2^L$ can be readily evaluated for a general contour (and for general parametrization), thanks to the factorized form (2.2) of the spinor couplings and to the identity (B.20). We have

$$(\eta_{L1} \gamma^\mu \bar{\eta}_2^L) = -2 \frac{(n_1 \cdot \bar{n}_2)}{(\eta_2 \bar{\eta}_1)} \left[\frac{\dot{x}_1^\mu}{|\dot{x}_1|} + \frac{\dot{x}_2^\mu}{|\dot{x}_2|} + \frac{1}{2} \frac{\dot{x}_1^\lambda \dot{x}_2^\nu}{|\dot{x}_1| |\dot{x}_2|} i\epsilon^{\lambda\mu\nu} \right]. \quad (4.6)$$

Since \dot{x}_1 , \dot{x}_2 and $x_{12} = -x_{21}$ lay on the same plane, we can drop the wedge product in (4.6) and we obtain the following result

$$\langle \text{Tr}_{\mathbf{N}} [(\eta\bar{\psi})_1(\psi\bar{\eta})_2] \rangle = -\frac{iMN\Gamma(1/2 - \epsilon)}{2\pi^{3/2-\epsilon}(\eta_2 \bar{\eta}_1)} (n_{L1} \bar{n}_2^L) \left[\frac{\dot{x}_1^\mu}{|\dot{x}_1|} + \frac{\dot{x}_2^\mu}{|\dot{x}_2|} \right] \partial_{x_1^\mu} \left(\frac{1}{(x_{12}^2)^{1/2-\epsilon}} \right), \quad (4.7)$$

which holds for any planar circuit. Let us specialize (4.4) to the diagram of 3.(e): within our parametrization of the circuit, we can rearrange the effective propagator exchanged through the rays as the difference of two total derivatives

$$\left[\frac{\dot{x}_1^\mu}{|\dot{x}_1|} + \frac{\dot{x}_2^\mu}{|\dot{x}_2|} \right] \partial_{x_1^\mu} \left(\frac{1}{(x_{12}^2)^{1/2-\epsilon}} \right) = \left(\frac{d}{d\tau_1} - \frac{d}{d\tau_2} \right) \frac{1}{(\tau_1^2 + \tau_2^2 - 2\tau_1\tau_2 \cos \varphi)^{1/2-\epsilon}}. \quad (4.8)$$

The integration over the two edges can be done in a rather trivial way

$$\begin{aligned} & \int_0^L d\tau_1 \int_{-L}^0 d\tau_2 \left(\frac{d}{d\tau_1} - \frac{d}{d\tau_2} \right) \frac{1}{(\tau_1^2 + \tau_2^2 - 2\tau_1\tau_2 \cos \varphi)^{1/2-\epsilon}} = \\ & = -\frac{L^{2\epsilon}}{\epsilon} + 2L^{2\epsilon} \int_0^1 d\tau \frac{1}{(\tau^2 + 2\tau \cos \varphi + 1)^{1/2-\epsilon}}. \end{aligned} \quad (4.9)$$

The remaining integral in (4.9) is finite as $\epsilon \rightarrow 0$ and it can be evaluated in terms of hypergeometric functions. However its exact value for arbitrary ϵ will not be relevant for us and we shall only give its expansion around $\epsilon = 0$ at the lowest order

$$\int_0^1 d\tau \frac{1}{(\tau^2 + 2\tau \cos \varphi + 1)^{1/2-\epsilon}} = \log \left(\sec \left(\frac{\varphi}{2} \right) + 1 \right) + O(\epsilon). \quad (4.10)$$

We end up with

$$\mathcal{F}_e = \frac{iMN\Gamma(1/2 - \epsilon)}{2\pi^{3/2-\epsilon}(\nu_1 \bar{\nu}_2)} (n_{L2} \bar{n}_1^L) L^{2\epsilon} \left[\frac{1}{\epsilon} - 2 \log \left(\sec \left(\frac{\varphi}{2} \right) + 1 \right) \right] \quad (4.11)$$

and since $(n_{L2} \bar{n}_1^L) = \cos \frac{\theta}{2}$ and $(\nu_1 \bar{\nu}_2) = 2i \cos \frac{\varphi}{2}$ we get

$$\mathcal{F}_e = MN \left(\frac{\Gamma(1/2 - \epsilon)}{4\pi^{3/2-\epsilon}} \right) \frac{\cos \frac{\theta}{2}}{\cos \frac{\varphi}{2}} L^{2\epsilon} \left[\frac{1}{\epsilon} - 2 \log \left(\sec \left(\frac{\varphi}{2} \right) + 1 \right) \right]. \quad (4.12)$$

Next we must consider the case where the fermionic propagator connects two points on the same edge of the cusp, *i.e.* the diagrams (b) in fig. 3. We have two mirror graphs: one for each edge. The result of the first one is provided by

$$-\frac{MN\Gamma(1/2-\epsilon)}{4\pi^{3/2-\epsilon}} \int_{-L}^0 d\tau_1 \int_{-L}^{\tau_1} d\tau_2 \left(\frac{d}{d\tau_1} - \frac{d}{d\tau_2} \right) \frac{1}{(\tau_1 - \tau_2)^{1-2\epsilon}} = -\frac{MN\Gamma(1/2-\epsilon)}{4\pi^{3/2-\epsilon}} \frac{L^{2\epsilon}}{\epsilon}, \quad (4.13)$$

while the contribution of the second one simply doubles (4.13) and it yields

$$\mathcal{F}_b = -2 \frac{MN\Gamma(1/2-\epsilon)}{4\pi^{3/2-\epsilon}} \frac{L^{2\epsilon}}{\epsilon}. \quad (4.14)$$

Therefore the complete one loop result for the upper left block can be written as

$$\mathfrak{F}^{(1)} = \left(\frac{2\pi}{\kappa} \right) MN \left(\frac{\Gamma(1/2-\epsilon)}{4\pi^{3/2-\epsilon}} \right) (\mu L)^{2\epsilon} \left[\frac{1}{\epsilon} \left(\frac{\cos \frac{\theta}{2}}{\cos \frac{\varphi}{2}} - 2 \right) - 2 \frac{\cos \frac{\theta}{2}}{\cos \frac{\varphi}{2}} \log \left(\sec \left(\frac{\varphi}{2} \right) + 1 \right) \right]. \quad (4.15)$$

This result may appear surprising at a first sight: while we could have expected the divergence from the cusp diagram (e), we have also a non-trivial contribution from the propagators living on a single edge (b). In $\mathcal{N} = 4$ SYM theory the analogous contributions, coming from the combined gauge-scalar propagator, are identically zero in Feynman gauge, and their potential divergence never enters into the game. Moreover in the limit $\varphi = \theta = 0$ a non-vanishing and divergent result persists, contradicting the naive expectation that the BPS infinite line is trivial. To understand the result (4.15) and to extract from it the truly gauge-invariant cusp divergence, we have to recall some basics about the renormalization of (cusped) Wilson loops in gauge theories and to adapt the general procedure to our somehow exotic operators: this will be done in the next section, after having completed the two-loop computation.

The full one-loop expression is recovered by considering also the part coming from the lower $M \times M$ block of the super-holonomy: it turns out to be the same, because of the symmetry between N and M at this order. The trace is simply obtained by adding this second contribution.

5 Two-loop analysis

We shall compute here the second order contribution to the expectation value of the cusped Wilson loop: we separate the computations of purely bosonic diagrams from fermionic ones, to appreciate technical and conceptual differences.

5.1 Bosonic diagrams

When expanding the Wilson loop operator at the second order in the coupling constant, we encounter the four families of merely bosonic contributions depicted in fig. 4. We consider first the diagrams containing the one-loop corrected gluon propagators (fig. 4.(a)). As we

did in the one-loop analysis, we shall focus our attention on the upper diagonal block of the super-matrix, *i.e.* on the $U(N)$ sector. With the help of (B.6), where the one-loop propagator is presented, we can immediately write

$$[4.(a)]_{\text{up}} = -MN^2 \left(\frac{2\pi}{\kappa} \right)^2 \frac{\Gamma^2(\frac{1}{2} - \epsilon)}{4\pi^{3-2\epsilon}} \int_{\Gamma} d\tau_{1>2} \left[\frac{(\dot{x}_1 \cdot \dot{x}_2)}{((x-y)^2)^{1-2\epsilon}} - \partial_{\tau_1} \partial_{\tau_2} \frac{((x_1 - x_2)^2)^{\epsilon}}{4\epsilon(1+2\epsilon)} \right]. \quad (5.1)$$

A similar structure is obtained when considering the correlator of two scalar composite operators $M_I^J C_J \bar{C}^I$ in the diagram 4.(b):

$$[4.(b)]_{\text{up}} = MN^2 \left(\frac{2\pi}{\kappa} \right)^2 \frac{\Gamma^2(\frac{1}{2} - \epsilon)}{16\pi^{3-2\epsilon}} \int_{\Gamma} d\tau_{1>2} \frac{|\dot{x}_1| |\dot{x}_2| \text{Tr}(M_1 M_2)}{((x-y)^2)^{1-2\epsilon}}. \quad (5.2)$$

The integrals (5.1) and (5.2) can be naturally combined together to give

$$-MN^2 \left(\frac{2\pi}{\kappa} \right)^2 \frac{\Gamma^2(\frac{1}{2} - \epsilon)}{4\pi^{3-2\epsilon}} \int_{\Gamma} d\tau_{1>2} \left[\frac{(\dot{x}_1 \cdot \dot{x}_2) - \frac{1}{4} |\dot{x}_1| |\dot{x}_2| \text{Tr}(M_1 M_2)}{((x-y)^2)^{1-2\epsilon}} - \partial_{\tau_1} \partial_{\tau_2} \frac{((x_1 - x_2)^2)^{\epsilon}}{4\epsilon(1+2\epsilon)} \right]. \quad (5.3)$$

The result (5.3) deserves some comments. The last term in (5.3) is a total derivative

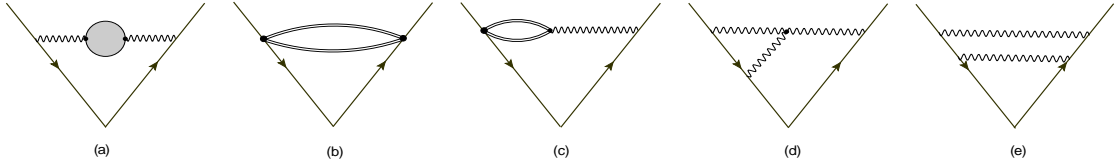


Figure 4. Two loops bosonic diagrams: (a) One-loop corrected gauge propagators; (b) Correlators of two composite scalar operators; (c) Correlators gauge field composite scalar operator; (d) Chern-Simons vertex diagrams; (e) Gluon double exchange diagrams.

and it would correspond to a gauge transformation -albeit a singular one. In dimensional regularization it yields a (θ, φ) independent pole in ϵ plus finite terms, thus its contribution to the divergent part of the cusp becomes ineffective when we impose the renormalization condition discussed in subsec. 4.1. The other contribution in (5.3), as firstly noted in [31], possesses an unforeseen four-dimensional structure. When the two endpoints lie on the same edge it is proportional to the tree-level effective propagator in $\mathcal{N} = 4$ since $\text{Tr}(M_1 M_2) = 4$ and thus it vanishes. If they lie instead on opposite edges we get the following result

$$\mathfrak{B}^{(2)} = -MN^2 \left(\frac{2\pi}{\kappa} \right)^2 \frac{\Gamma^2(\frac{1}{2} - \epsilon)}{4\pi^{3-2\epsilon}} \left(\cos \varphi - \cos^2 \frac{\theta}{2} \right) \int_0^L d\tau_1 \int_{-L}^0 d\tau_2 \frac{1}{((x-y)^2)^{1-2\epsilon}}, \quad (5.4)$$

where the integral governing the divergence is the same of the four dimensional case when we replace 2ϵ with ϵ .

Next we examine the graphs 4.(c), 4.(d) and 4.(e). The last one is identically zero for the same reasons of the one-loop single exchange 3.(a). The diagram 4.(c) for the case of planar loop was discussed in [31] where it was found to vanish. The same fate is shared by 4.(d) as pointed out in [26]. The only contribution originating from the bosonic diagrams is therefore provided by (5.4).

5.2 Fermionic diagrams

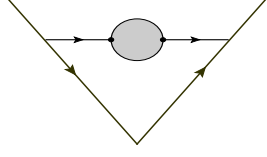


Figure 5. One-loop corrected fermions propagators

The simplest fermionic diagram appearing at the second order in perturbation theory consists of the exchange of the one-loop corrected fermion propagator depicted in fig 5.

The one-loop two-point function for the spinor fields is briefly discussed in app. B. Remarkably it again displays the four dimensional behaviour already encountered in the bosonic case. Its form, in the DRED scheme, is

$$\left\langle (\psi_I)_i^j(x) (\bar{\psi}^J)_k^{\hat{l}}(y) \right\rangle_0^{1\text{ loop}} = -i \left(\frac{2\pi}{\kappa} \right) \delta_i^{\hat{l}} \delta_k^j (N - M) \frac{\Gamma^2(\frac{1}{2} - \epsilon)}{16\pi^{3-2\epsilon}} \frac{1}{((x - y)^2)^{1-2\epsilon}}. \quad (5.5)$$

The contribution to the upper block of the Wilson loop takes the following form

$$\left(\frac{2\pi\mu^{2\epsilon}}{\kappa} \right)^2 iMN(N - M) \frac{\Gamma^2(\frac{1}{2} - \epsilon)}{16\pi^{3-2\epsilon}} \int_{\Gamma} d\tau_{1>2} \frac{(\eta_{I1}\bar{\eta}_2^I)}{((x_1 - x_2)^2)^{1-2\epsilon}} \quad (5.6)$$

5.2.1 Double Exchanges

We come now to discuss a more subtle group of diagrams, namely those involving two $\langle \psi \bar{\psi} \rangle$ propagators. They arise when we evaluate the contribution of the fermionic quadrilinear in (4.2). At this order its expansion yields only two sets of non-vanishing Wick-contractions, weighted by different group factor, and thus we arrive at the following integral

$$-4 \left(\frac{2\pi\mu^{2\epsilon}}{\kappa} \right)^2 \int_{\Gamma} d\tau_{1>2>3>4} [M^2 N \underset{(A1)}{S(x_2 - x_1)S(x_4 - x_3)} - N^2 M \underset{(B1)}{S(x_2 - x_3)S(x_4 - x_1)}]. \quad (5.7)$$

Here the function $S(x_i - x_j)$ is proportional to the two-point fermion correlator already encountered in (4.5) and it can be conveniently written as

$$S(x_i - x_j) = \frac{(n_i \cdot n_j)}{(\eta_j \bar{\eta}_i)} (\partial_{\tau_i} - \partial_{\tau_j}) D(x_i - x_j), \quad (5.8)$$

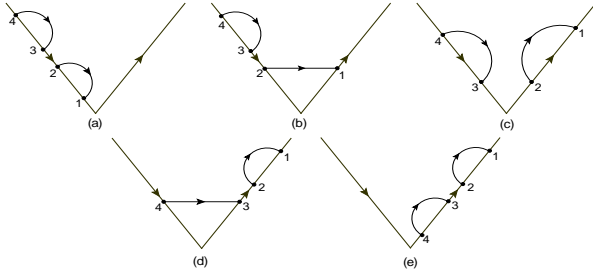


Figure 6. First group of double-exchange diagrams

ent Feynman diagrams depicted in fig. 6. Luckily we do not have to compute all of them.

where $D(x_i - x_j)$ is the free scalar propagator defined in (B.3) while the couple of vectors (n_{iI}, \bar{n}_i^I) and of spinors $(\eta_{iI}, \bar{\eta}_i^I)$ are defined in sec. 3.1. We shall consider the two contributions in (5.7) separately. In order to evaluate the term (A1) we have to split the region of integration in five sectors that correspond to the five different

In fact graphs, which are related by a reflection with respect to the axis bisecting the cusp, yield the same result¹¹. In other words, the following equalities hold among the diagrams of fig.6: 6.(a)=6.(e) and 6.(b)=6.(d). Moreover the graph 6.(c) is simply the square of 3.(b).

To begin with, let us evaluate the contribution 6.(a). It is given by the following integral

$$\begin{aligned} [6.(a)]_{\text{up}} &= \left(\frac{2\pi}{\kappa}\right)^2 M^2 N \frac{\Gamma^2\left(\frac{3}{2} - \epsilon\right)}{\pi^{3-2\epsilon}} \mu^{4\epsilon} \int_{-L}^0 d\tau_1 \int_{-L}^{\tau_1} d\tau_2 \int_{-L}^{\tau_2} d\tau_3 \int_{-L}^{\tau_3} d\tau_4 (\tau_1 - \tau_2)^{2\epsilon-2} (\tau_3 - \tau_4)^{2\epsilon-2} = \\ &= \left(\frac{2\pi}{\kappa}\right)^2 M^2 N \frac{\Gamma^2\left(\frac{1}{2} - \epsilon\right)}{16\pi^{3-2\epsilon}} \frac{\sqrt{\pi}}{2^{4\epsilon}} \frac{\Gamma(2\epsilon+1)}{\Gamma(2\epsilon+\frac{1}{2})} \frac{(\mu L)^{4\epsilon}}{\epsilon^2}. \end{aligned} \quad (5.9)$$

The diagram 6.(b) instead leads to a different computation

$$[6.(b)]_{\text{up}} = - \left(\frac{2\pi}{\kappa}\right)^2 M^2 N \frac{\Gamma^2\left(\frac{1}{2} - \epsilon\right)}{16\pi^{3-2\epsilon}} \frac{\mu^{4\epsilon}}{\epsilon} \frac{\cos \frac{\theta}{2}}{\cos \frac{\varphi}{2}} \int_0^L d\tau_1 \int_{-L}^0 d\tau_2 (L + \tau_2)^{2\epsilon} (\partial_{\tau_2} - \partial_{\tau_1}) H(\tau_1, \tau_2), \quad (5.10)$$

where $H(\tau_1, \tau_2) = (\tau_1^2 + \tau_2^2 - 2\tau_1\tau_2 \cos \varphi)^{-\frac{1}{2}+\epsilon}$. We have performed the two trivial integrations over τ_3 and τ_4 since they involve a propagator whose endpoints belongs to the same edge. To extract the result we are interested in, we do not need the exact value of the remaining integral, but only its ϵ -expansion up to finite terms discussed in app. C. We get

$$[6.(b)]_{\text{up}} = - \left(\frac{2\pi}{\kappa}\right)^2 M^2 N \frac{\cos \frac{\theta}{2}}{\cos \frac{\varphi}{2}} \frac{\Gamma^2\left(\frac{1}{2} - \epsilon\right)}{16\pi^{3-2\epsilon}} (L\mu)^{4\epsilon} \left[\frac{1}{\epsilon^2} - \frac{2}{\epsilon} \log \left(1 + \sec \frac{\varphi}{2}\right) + O(1) \right]. \quad (5.11)$$

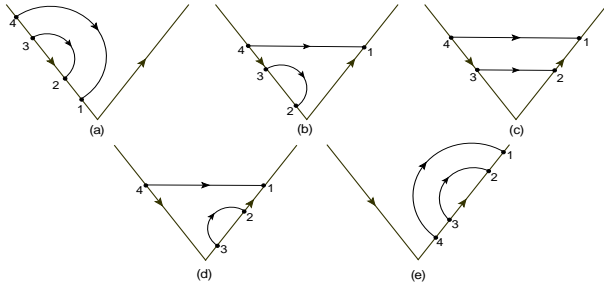


Figure 7. Second group of double-exchange diagrams

The next step is to consider the term (B1) in (5.7): again we have to separate the region of integration in five sub-sectors and this yields the diagrams in fig. 7. However the same reflection symmetry considered in the case of the term (A1) implies that we have just to compute 7.(a), 7.(b) and 7.(c). The first one can be easily computed in closed form and it gives

$$[7.(a)]_{\text{up}} = \left(\frac{2\pi}{\kappa}\right)^2 M N^2 \frac{\Gamma\left(\frac{1}{2} - \epsilon\right)^2}{16\pi^{3-2\epsilon}} \frac{(2\epsilon-1)}{2(4\epsilon-1)} \frac{(\mu L)^{4\epsilon}}{\epsilon^2}. \quad (5.12)$$

Concerning the second diagram, we can trivially perform the integration over τ_2 and τ_3 , obtaining

$$[7.(b)]_{\text{up}} = - \left(\frac{2\pi}{\kappa}\right)^2 N^2 M \frac{\Gamma^2\left(\frac{1}{2} - \epsilon\right)}{16\pi^{3-2\epsilon}} \frac{\cos \frac{\theta}{2}}{\cos \frac{\varphi}{2}} \frac{\mu^{4\epsilon}}{\epsilon} \int_0^L d\tau_1 \int_{-L}^0 d\tau_4 (-\tau_4)^{2\epsilon} (\partial_{\tau_4} - \partial_{\tau_1}) H(\tau_1, \tau_4). \quad (5.13)$$

¹¹This equality can be shown by performing the change of variable $s_i \mapsto -s_{5-i}$ ($i = 1, \dots, 4$) and subsequently by restoring the integration in the canonical order.

With the help of the results of app. C, we can then write the following ϵ -expansion

$$[7.(b)]_{\text{up}} = - \left(\frac{2\pi}{\kappa} \right)^2 N^2 M \frac{\Gamma^2(\frac{1}{2} - \epsilon)}{16\pi^{3-2\epsilon}} \frac{\cos \frac{\theta}{2}}{\cos \frac{\varphi}{2}} (L\mu)^{4\epsilon} \left[\frac{1}{2\epsilon^2} + \frac{1}{\epsilon} \log \left(\frac{1}{4} \cos \frac{\varphi}{2} \sec^4 \frac{\varphi}{4} \right) + O(1) \right]. \quad (5.14)$$

For the graph 7.(c) we shall adopt a different procedure since both propagators connect different edges. First we rearrange its integral expression as follows

$$[7.(c)]_{\text{up}} = -2 \left(\frac{2\pi}{\kappa} \right)^2 N^2 M \left(\mu^{2\epsilon} \int_0^L d\tau_1 \int_{-L}^0 d\tau_4 S(x_4 - x_1) \right)^2 - \\ - 4 \left(\frac{2\pi}{\kappa} \right)^2 M N^2 \mu^{4\epsilon} \int_0^L d\tau_1 \int_0^{\tau_1} d\tau_2 \int_{-L}^0 d\tau_3 \int_{-L}^{\tau_3} S(x_2 - x_4) S(x_3 - x_1), \quad (5.15)$$

where we have separated the “abelian” and “non-abelian” part of the diagram. The former is given by the first term, which is proportional to the square of 3.(e), while the latter is identified with the second term in (5.15). This decomposition also has advantage that the leading divergence $1/\epsilon^2$ is only present in the first term.

With the help of the results of app. C, we obtain the following ϵ -expansion for this diagram

$$[7.(c)]_{\text{up}} = \frac{1}{2} \left(\frac{2\pi}{\kappa} \right)^2 N^2 M \frac{\Gamma^2(\frac{1}{2} - \epsilon)}{16\pi^{3-2\epsilon}} (L\mu)^{4\epsilon} \left[\frac{\cos \frac{\theta}{2}}{\cos \frac{\varphi}{2}} \left(\frac{1}{\epsilon} - 2 \log \left(1 + \sec \frac{\varphi}{2} \right) \right) \right]^2 - \\ - \left(\frac{2\pi}{\kappa} \right)^2 N^2 M \left(\frac{\cos \frac{\theta}{2}}{\cos \frac{\varphi}{2}} \right)^2 \frac{\Gamma^2(\frac{1}{2} - \epsilon)}{16\pi^{3-2\epsilon}} \frac{(L\mu)^{4\epsilon}}{\epsilon} \cos^2 \frac{\varphi}{2} \frac{\varphi}{\sin \varphi} + O(1). \quad (5.16)$$

5.2.2 Vertex Diagrams

The final group of fermionic diagrams, which are relevant for our calculation, arises when we expand in perturbation theory the term

$$- \frac{2\pi i}{\kappa} \int_{\Gamma} d\tau_{1>2>3} \left\langle \begin{matrix} (\eta\bar{\psi})_1(\psi\bar{\eta})_2 \mathcal{A}_3 \\ (\mathbf{A}_2) \end{matrix} + \begin{matrix} \mathcal{A}_1(\eta\bar{\psi})_2(\psi\bar{\eta})_3 \\ (\mathbf{B}_2) \end{matrix} + \begin{matrix} (\eta\bar{\psi})_1 \hat{\mathcal{A}}_2(\psi\bar{\eta})_3 \\ (\mathbf{C}_2) \end{matrix} \right\rangle, \quad (5.17)$$

appearing in the upper block (4.2). At this order the expectation value in (5.17) is evaluated by just considering the Wick-contractions of the monomials (\mathbf{A}_2) , (\mathbf{B}_2) and (\mathbf{C}_2) with the tree-level gauge-fermion vertices present in the Lagrangian (A.1). Then the three different contributions can be rewritten as follows

$$(\mathbf{A}_2) = - \left(\frac{2\pi}{\kappa} \right)^2 N^2 M \int_{\Gamma} d\tau_{1>2>3} \eta_{1L} \gamma_{\nu} \gamma^{\mu} \gamma_{\lambda} \bar{\eta}_2^L \epsilon_{\mu\rho\sigma} \dot{x}_3^{\rho} \Gamma^{\nu\lambda\sigma}(x_1, x_2, x_3), \quad (5.18a)$$

$$(\mathbf{B}_2) = - \left(\frac{2\pi}{\kappa} \right)^2 N^2 M \int_{\Gamma} d\tau_{1>2>3} \eta_{2L} \gamma_{\lambda} \gamma^{\mu} \gamma_{\nu} \bar{\eta}_3^L \epsilon_{\mu\rho\sigma} \dot{x}_1^{\rho} \Gamma^{\sigma\lambda\nu}(x_1, x_2, x_3), \quad (5.18b)$$

$$(\mathbf{C}_2) = - \left(\frac{2\pi}{\kappa} \right)^2 N M^2 \int_{\Gamma} d\tau_{1>2>3} \eta_{1L} \gamma_{\lambda} \gamma^{\mu} \gamma_{\nu} \bar{\eta}_3^L \epsilon_{\mu\rho\sigma} \dot{x}_2^{\rho} \Gamma^{\lambda\sigma\nu}(x_1, x_2, x_3), \quad (5.18c)$$

where $\Gamma^{\lambda\mu\nu}(x_1, x_2, x_3)$ is a short-hand notation for the three-point function in position space, defined by the integral

$$\Gamma^{\lambda\mu\nu}(x_1, x_2, x_3) = \left(\frac{\Gamma(\frac{1}{2} - \epsilon)}{4\pi^{3/2-\epsilon}} \right)^3 \partial_{x_1^\lambda} \partial_{x_2^\mu} \partial_{x_3^\nu} \int \frac{d^{3-2\epsilon}w}{(x_{1w}^2)^{1/2-\epsilon} (x_{2w}^2)^{1/2-\epsilon} (x_{3w}^2)^{1/2-\epsilon}}. \quad (5.19)$$

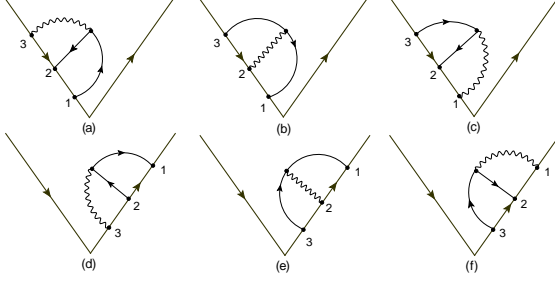


Figure 8. Vertex diagrams where the fermion propagators are attached to the same edge of the gluon propagator.

For instance, 8.(a) and 8.(d) originate from (\mathbf{A}_2) , 8.(b) and 8.(e) from (\mathbf{C}_2) and 8.(c) and 8.(f) from (\mathbf{B}_2) .

If we now expand the spinor bilinears in (5.18) in terms of the circuit tangent vectors \dot{x}_i and of the scalar contraction $(\eta_i \bar{\eta}_j)$, the three contributions (\mathbf{A}_2) , (\mathbf{B}_2) and (\mathbf{C}_2) can be rewritten as follows

$$\begin{aligned} (\mathbf{A}_2) = & -N^2 M \left(\frac{2\pi}{\kappa} \right)^2 i(n_1 \cdot n_2) \oint_{\tau_1 > \tau_2 > \tau_3} \left[\frac{2}{(\eta_2 \bar{\eta}_1)} \left(((\dot{x}_2 \cdot \dot{x}_3) \dot{x}_{1\nu} - (\dot{x}_1 \cdot \dot{x}_3) \dot{x}_{2\nu}) [\Gamma^{\nu\tau\tau} + \right. \right. \\ & + \Gamma^{\tau\nu\tau} - \Gamma^{\tau\tau\nu}] + \dot{x}_{1\sigma} \dot{x}_{2\nu} \dot{x}_{3\lambda} \Gamma^{\nu\lambda\sigma} - \dot{x}_{1\nu} \dot{x}_{2\sigma} \dot{x}_{3\lambda} \Gamma^{\nu\lambda\sigma} + \dot{x}_{1\sigma} \dot{x}_{2\lambda} \dot{x}_{3\nu} \Gamma^{\nu\lambda\sigma} - \\ & \left. \left. - \dot{x}_{1\lambda} \dot{x}_{2\sigma} \dot{x}_{3\nu} \Gamma^{\nu\lambda\sigma} \right) + (\eta_1 \bar{\eta}_2) \dot{x}_{3\lambda} (\Gamma^{\tau\lambda\tau} - \Gamma^{\lambda\tau\tau}) \right], \end{aligned} \quad (5.20a)$$

$$\begin{aligned} (\mathbf{B}_2) = & -N^2 M \left(\frac{2\pi}{\kappa} \right)^2 i(n_3 \cdot n_2) \oint_{\tau_1 > \tau_2 > \tau_3} \left[\frac{2}{(\eta_3 \bar{\eta}_2)} \left([(\dot{x}_1 \cdot \dot{x}_3) \dot{x}_{2\nu} - (\dot{x}_2 \cdot \dot{x}_1) \dot{x}_{3\nu}] [\Gamma^{\tau\tau\nu} + \right. \right. \\ & + \Gamma^{\tau\nu\tau} - \Gamma^{\nu\tau\tau}] + \dot{x}_{1\lambda} \dot{x}_{2\nu} \dot{x}_{3\sigma} \Gamma^{\nu\lambda\sigma} - \dot{x}_{1\lambda} \dot{x}_{2\sigma} \dot{x}_{3\nu} \Gamma^{\nu\lambda\sigma} + \dot{x}_{1\sigma} \dot{x}_{2\nu} \dot{x}_{3\lambda} \Gamma^{\nu\lambda\sigma} - \\ & \left. \left. - \dot{x}_{1\sigma} \dot{x}_{2\lambda} \dot{x}_{3\nu} \Gamma^{\nu\lambda\sigma} \right) + (\eta_2 \bar{\eta}_3) \dot{x}_{1\nu} (\Gamma^{\tau\tau\nu} - \Gamma^{\tau\nu\tau}) \right], \end{aligned} \quad (5.20b)$$

Diagrammatically the three contributions (5.18) will lead to graphs which differ for the position of the gauge field along the contour: (\mathbf{A}_2) and (\mathbf{B}_2) only yield diagrams where the gluon is respectively the first or the last field we encounter when τ runs from $-L$ to L ; (\mathbf{C}_2) instead corresponds to diagrams where the gauge field is always located between the two fermionic

$$\begin{aligned}
(\mathbf{C}_2) = & -NM^2 \left(\frac{2\pi}{\kappa} \right)^2 i(n_1 \cdot n_3) \oint_{\tau_1 > \tau_2 > \tau_3} \left[\frac{2}{(\eta_3 \bar{\eta}_1)} \left(((\dot{x}_2 \cdot \dot{x}_3) \dot{x}_{1\nu} - (\dot{x}_1 \cdot \dot{x}_2) \dot{x}_{3\nu}) [\Gamma^{\tau\tau\nu} + \right. \right. \\
& + \Gamma^{\nu\tau\tau} - \Gamma^{\tau\nu\tau}] + \dot{x}_{1\sigma} \dot{x}_{3\nu} \dot{x}_{2\lambda} \Gamma^{\lambda\sigma\nu} - \dot{x}_{1\nu} \dot{x}_{3\sigma} \dot{x}_{2\lambda} \Gamma^{\lambda\sigma\nu} + \dot{x}_{1\sigma} \dot{x}_{3\lambda} \dot{x}_{2\nu} \Gamma^{\lambda\sigma\nu} - \\
& \left. \left. - \dot{x}_{1\lambda} \dot{x}_{3\sigma} \dot{x}_{2\nu} \Gamma^{\lambda\sigma\nu} \right) + (\eta_1 \bar{\eta}_3) \dot{x}_{2\nu} (\Gamma^{\tau\tau\nu} - \Gamma^{\nu\tau\tau}) \right], \tag{5.20c}
\end{aligned}$$

where we have dropped all the terms which vanish for planar contours. To begin with, we shall consider the family of diagrams of fig. 8, where all the bosonic and fermionic lines terminate on the same edge of the cusp. In this case all the terms proportional to the factor $2/(\eta_i \bar{\eta}_j)$ in (5.20) drop out because the tangent vectors obey the relation

$$\dot{x}_1 = \dot{x}_2 = \dot{x}_3, \tag{5.21}$$

for each diagram in fig. 8. Only the last terms in (5.20a), (5.20b) and (5.20c) that are proportional to the bilinear $(\eta_i \bar{\eta}_j)$ are different from zero and we are left with

$$(\mathbf{A}_2) = 2N^2 M \left(\frac{2\pi}{\kappa} \right)^2 \oint_{\tau_1 > \tau_2 > \tau_3} \dot{x}_{3\lambda} (\Gamma^{\tau\lambda\tau} - \Gamma^{\lambda\tau\tau}) \equiv - \left(\frac{2\pi}{\kappa} \right)^2 N^2 M(\mathbf{a}) \tag{5.22a}$$

$$(\mathbf{B}_2) = 2N^2 M \left(\frac{2\pi}{\kappa} \right)^2 \oint_{\tau_1 > \tau_2 > \tau_3} \dot{x}_{1\nu} (\Gamma^{\tau\tau\nu} - \Gamma^{\tau\nu\tau}) \equiv - \left(\frac{2\pi}{\kappa} \right)^2 N^2 M(\mathbf{b}) \tag{5.22b}$$

$$(\mathbf{C}_2) = 2NM^2 \left(\frac{2\pi}{\kappa} \right)^2 \oint_{\tau_1 > \tau_2 > \tau_3} \dot{x}_{2\nu} (\Gamma^{\tau\tau\nu} - \Gamma^{\nu\tau\tau}) \equiv - \left(\frac{2\pi}{\kappa} \right)^2 NM^2(\mathbf{c}), \tag{5.22c}$$

where we used that $\eta_i \bar{\eta}_j = 2i$ and $(n_i \cdot n_j) = 1$. There is a further simplification: in fact we do not have to compute all the diagrams originating from (\mathbf{A}_2) , (\mathbf{B}_2) and (\mathbf{C}_2) and depicted in fig. 8. First of all, we can restrict ourselves to considering only the diagrams 8.(a), 8.(b) and 8.(c). The other three graphs will simply double the final result. Next, we note that the following identity holds for this subclass of diagrams

$$(\mathbf{a}) + (\mathbf{b}) = (\mathbf{c}), \tag{5.23}$$

i.e. it is sufficient to evaluate only the integral

$$(\mathbf{c}) = -2 \int_{-L}^0 d\tau_1 \int_{-L}^{\tau_1} d\tau_2 \int_{-L}^{\tau_2} d\tau_3 \dot{x}_{2\nu} (\Gamma^{\tau\tau\nu} - \Gamma^{\nu\tau\tau}) \tag{5.24}$$

to reconstruct the result of all the diagrams in fig. 8. Moreover the three-point functions appearing in (5.24) always possess two contracted indices: in this case the integral (5.19) can be easily evaluated in terms of product of scalar propagators and one finds

$$\Gamma^{\tau\tau\nu} = \partial_{x_3^\nu} \Phi_{3,12}, \quad \Gamma^{\tau\nu\tau} = \partial_{x_2^\nu} \Phi_{2,13}, \quad \Gamma^{\nu\tau\tau} = \partial_{x_1^\nu} \Phi_{1,23}, \tag{5.25}$$

where

$$\Phi_{i,jk} = - \frac{\Gamma^2(1/2 - \epsilon)}{32\pi^{3-2\epsilon}} \left[\frac{1}{(x_{ij}^2)^{\frac{1}{2}-\epsilon} (x_{ik}^2)^{\frac{1}{2}-\epsilon}} - \frac{1}{(x_{ij}^2)^{\frac{1}{2}-\epsilon} (x_{kj}^2)^{\frac{1}{2}-\epsilon}} - \frac{1}{(x_{ik}^2)^{\frac{1}{2}-\epsilon} (x_{jk}^2)^{\frac{1}{2}-\epsilon}} \right]. \tag{5.26}$$

See appendix C for more details. With the help of this result, and recalling (5.21), we can show that the integrand in (5.24) only contains total derivatives and can be easily computed

$$\begin{aligned}
(c) &= -2 \int_{-L}^0 d\tau_1 \int_{-L}^{\tau_1} d\tau_2 \int_{-L}^{\tau_2} d\tau_3 \left(\frac{d}{d\tau_3} \Phi_{3,12} - \frac{d}{d\tau_1} \Phi_{1,23} \right) = \\
&= -2 \int_{-L}^0 d\tau_1 \int_{-L}^{\tau_1} d\tau_2 (\Phi_{2,12} + \Phi_{1,12} - \Phi_{-L,12} - \Phi_{0,12}) = \\
&= 2 \frac{\Gamma^2(1/2 - \epsilon)}{32\pi^{3-2\epsilon}} L^{4\epsilon} \left(\frac{1}{2\epsilon^2} + \frac{1}{2\epsilon} + O(1) \right). \tag{5.27}
\end{aligned}$$

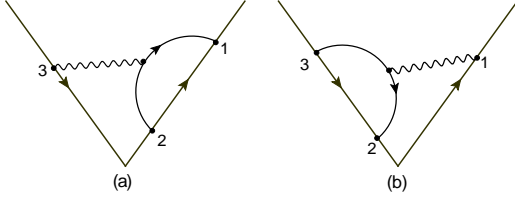


Figure 9. Vertex diagrams with fermionic propagators attached to the opposite edge of the gluon propagator.

Next we consider the case where the fermions are both attached to the same line, but the gluon is not. We have the two possibilities depicted in fig. 9. The diagram 9.(a) originates from the contribution (5.20a) when considering the region of integration $-L \leq \tau_3 \leq 0$ and $0 \leq \tau_2 \leq \tau_1 \leq L$.

The diagram 9.(b) is instead obtained from (5.20b), when $-L \leq \tau_3 \leq \tau_2 \leq 0$, $0 \leq \tau_1 \leq L$ and $\dot{x}_3 = \dot{x}_2$. No contribution of this kind is instead contained in (5.20c). Since the two graphs in fig. 9 are related by a reflection with respect to the axis bisecting the cusp, they yield the same result and thus we have to compute only one of them, *e.g.* 9.(a). For this diagram all the terms in (5.20a), which are not proportional to $(\eta_1 \bar{\eta}_2)$, will vanish when we use that $\dot{x}_1 = \dot{x}_2$ and so we get an expression that is similar to the one considered in (5.22a):

$$[9.(a)]_{\text{up}} = 2N^2 M \left(\frac{2\pi}{\kappa} \right)^2 \int_0^L d\tau_1 \int_0^{\tau_1} d\tau_2 \int_{-L}^0 d\tau_3 \dot{x}_{3\lambda} (\Gamma^{\tau\lambda\tau} - \Gamma^{\lambda\tau\tau}). \tag{5.28}$$

In order to compute this integral we first observe that the integrand can be rearranged as follows

$$\begin{aligned}
\dot{x}_{3\lambda} (\Gamma^{\tau\lambda\tau} - \Gamma^{\lambda\tau\tau}) &= \dot{x}_3 \cdot \partial_{x_2} \Phi_{2,13} - \dot{x}_3 \cdot \partial_{x_1} \Phi_{1,23} = \dot{x}_3 \cdot \partial_{x_2} (\Phi_{2,13} + \Phi_{1,23}) + \frac{d}{d\tau_3} \Phi_{1,23} \\
&= - \left(\frac{\Gamma(1/2 - \epsilon)}{4\pi^{3/2-\epsilon}} \right)^2 \frac{1}{(x_{13}^2)^{1/2-\epsilon}} \frac{d}{d\tau_3} \frac{1}{(x_{23}^2)^{1/2-\epsilon}} + \frac{d}{d\tau_3} \Phi_{1,23}. \tag{5.29}
\end{aligned}$$

We have two separate contributions, which both appear in the list considered in appendix C (see eqs. (C.26) and (C.28)) and thus we can immediately write the final result

$$[9.(a)]_{\text{up}} = -N^2 M \left(\frac{2\pi}{\kappa} \right)^2 L^{4\epsilon} \left(\frac{\Gamma(1/2 - \epsilon)}{4\pi^{3/2-\epsilon}} \right)^2 \left[\frac{1}{\epsilon} \left(\log \left(\cos \frac{\varphi}{2} \right) - \frac{1}{2} \varphi \cot \varphi \right) + O(1) \right]. \tag{5.30}$$

The final set of diagrams that we have to consider are those where the two fermions end on different edges of the cusp. We have four possible graphs of this kind, which simply

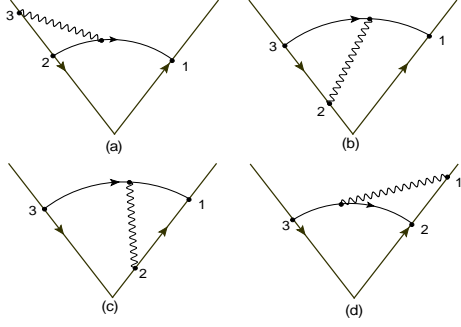


Figure 10. Vertex diagrams with fermionic propagators are attached to opposite edges.

parameters τ_i . Again graphs, which are related by a reflection with respect the axis bisecting the cusp, produce the same result and we focus our attention only on 10.(a) and 10.(b).

To begin with, we shall factor out from both diagrams the color and R -symmetry dependence and we shall write

$$[10.(a)]_{\text{up}} \equiv -N^2 M \left(\frac{2\pi}{\kappa} \right)^2 \cos \frac{\theta}{2} \mathcal{I}_{(a)} \quad [10.(b)]_{\text{up}} \equiv -NM^2 \left(\frac{2\pi}{\kappa} \right)^2 \cos \frac{\theta}{2} \mathcal{I}_{(b)}. \quad (5.31)$$

In order to simplify our analysis we shall construct the two independent combinations $\mathcal{I}_{(a)} - \mathcal{I}_{(b)}$ and $\mathcal{I}_{(a)} + \mathcal{I}_{(b)}$. The former is the only combination of the two integrals appearing in the final result when we would take the super-trace of the Wilson-loop and it is given by

$$\begin{aligned} \mathcal{I}_{(a)} - \mathcal{I}_{(b)} &= \frac{2i}{(\eta_2 \bar{\eta}_1)} \int_0^L d\tau_1 \int_{-L}^0 d\tau_2 \int_{-L}^{\tau_2} d\tau_3 \left(\dot{x}_{1\nu} - (\dot{x}_1 \cdot \dot{x}_3) \dot{x}_{2\nu} - \frac{(\eta_1 \bar{\eta}_2)(\eta_2 \bar{\eta}_1)}{2} \dot{x}_{3\nu} \right) (\Gamma^{\tau\tau\nu} - \Gamma^{\tau\nu\tau}) = \\ &= \frac{2i}{(\eta_2 \bar{\eta}_1)} \int_0^L d\tau_1 \int_{-L}^0 d\tau_2 \int_{-L}^{\tau_2} d\tau_3 (\dot{x}_{1\nu} + \dot{x}_{2\nu}) (\Gamma^{\tau\tau\nu} - \Gamma^{\tau\nu\tau}). \end{aligned} \quad (5.32)$$

In (5.32) we were able to get rid of all the terms containing a three-point function contracted with three \dot{x}_i , thanks to the identity (B.15) and to the equality $\dot{x}_2 = \dot{x}_3$, which holds for these diagrams. We can now use the relations (5.25) and the invariance under translation of the function $\Phi_{i,jk}$ to rewrite the integrand as follows

$$\begin{aligned} (\dot{x}_{1\nu} + \dot{x}_{2\nu}) (\Gamma^{\tau\tau\nu} - \Gamma^{\tau\nu\tau}) &= \frac{d}{d\tau_3} \Phi_{3,12} - \frac{d}{d\tau_2} \Phi_{2,13} - \frac{d}{d\tau_1} \Phi_{3,12} + \\ &+ \left(\frac{\Gamma(1/2 - \epsilon)}{4\pi^{3/2-\epsilon}} \right)^2 \frac{1}{(x_{13}^2)^{1/2-\epsilon}} \frac{d}{d\tau_1} \frac{1}{(x_{12}^2)^{1/2-\epsilon}}. \end{aligned} \quad (5.33)$$

The integration over the circuit can be performed by means of the results given in app. C and we find

$$\mathcal{I}_{(a)} - \mathcal{I}_{(b)} = \frac{L^{4\epsilon}}{2 \cos \frac{\varphi}{2}} \left(\frac{\Gamma(1/2 - \epsilon)}{4\pi^{3/2-\epsilon}} \right)^2 \left[\frac{1}{2\epsilon} \frac{\varphi}{\sin \varphi} + \frac{1}{\epsilon} \log \left(\cos \frac{\varphi}{2} \right) + \frac{1}{4\epsilon^2} - \right. \quad (5.34)$$

$$\left. - \frac{1}{\epsilon} \left(\frac{\varphi}{2} \cot \varphi - \log \left(\cos \frac{\varphi}{2} \right) \right) + \frac{1}{4\epsilon^2} + O(1) \right] = \quad (5.35)$$

differ for the position of the gluon line, and they are displayed in fig. 10. The diagrams 10.(a) and 10.(d) are obtained respectively from (5.20a) and (5.20b) when considering the region of integrations (I) = $\{-L \leq \tau_3 \leq \tau_2 \leq 0$ and $0 \leq \tau_1 \leq L\}$ and (II) = $\{-L \leq \tau_3 \leq 0$ and $0 \leq \tau_2 \leq \tau_1 \leq L\}$. The diagrams 10.(b) and 10.(c) originate instead from (5.20c) when choosing either the range (I) or (II) for the

$$= \left(\frac{\Gamma(1/2 - \epsilon)}{4\pi^{3/2-\epsilon}} \right)^2 \frac{L^{4\epsilon}}{4\epsilon} \frac{\varphi}{\sin \frac{\varphi}{2}} + O(1). \quad (5.36)$$

The sum $\mathcal{I}_{(a)} + \mathcal{I}_{(b)}$ is instead the only combination appearing in the final result if we take the trace of the loop operator. Its expression is less elegant than the one for the difference and it is given by

$$\begin{aligned} \mathcal{I}_{(a)} + \mathcal{I}_{(b)} = & i \int_0^L d\tau_1 \int_{-L}^0 d\tau_2 \int_{-L}^{\tau_2} d\tau_3 \left[\frac{2}{(\eta_2 \bar{\eta}_1)} \left(2(\dot{x}_{1\nu} + \dot{x}_{3\nu}) \Gamma^{\nu\tau\tau} + \dot{x}_{2\nu} \dot{x}_{1\lambda} \dot{x}_{3\sigma} \Gamma^{\nu\lambda\sigma} + \right. \right. \\ & \left. \left. + \dot{x}_{2\lambda} \dot{x}_{1\sigma} \dot{x}_{3\nu} \Gamma^{\nu\lambda\sigma} - 2\dot{x}_{1\nu} \dot{x}_{3\lambda} \dot{x}_{2\sigma} \Gamma^{\nu\lambda\sigma} \right) + (\eta_1 \bar{\eta}_2) \dot{x}_{3\nu} (\Gamma^{\tau\nu\tau} + \Gamma^{\tau\tau\nu}) \right]. \end{aligned} \quad (5.37)$$

It is not difficult to realize that the integrand in (5.37) is symmetric when exchanging τ_2 with τ_3 : this allows us to extend the integration over τ_3 up to 0 provided dividing the result by two. We can reorganize (5.37) as follows

$$\begin{aligned} \mathcal{I}_{(a)} + \mathcal{I}_{(b)} = & \frac{i}{2} \int_0^L d\tau_1 \int_{-L}^0 d\tau_2 \int_{-L}^0 d\tau_3 \left[\frac{2}{(\eta_2 \bar{\eta}_1)} \left(2(\dot{x}_{1\nu} + \dot{x}_{3\nu}) \Gamma^{\nu\tau\tau} + \dot{x}_{2\nu} \dot{x}_{1\lambda} \dot{x}_{3\sigma} \Gamma^{\nu\lambda\sigma} + \right. \right. \\ & \left. \left. + \dot{x}_{2\lambda} \dot{x}_{1\sigma} \dot{x}_{3\nu} \Gamma^{\nu\lambda\sigma} - 2\dot{x}_{1\nu} \dot{x}_{3\lambda} \dot{x}_{2\sigma} \Gamma^{\nu\lambda\sigma} \right) + (\eta_1 \bar{\eta}_2) \dot{x}_{3\nu} (\Gamma^{\tau\nu\tau} + \Gamma^{\tau\tau\nu}) \right] = \\ = & i \int_0^L d\tau_1 \int_{-L}^0 d\tau_2 \int_{-L}^0 d\tau_3 \left[\frac{2}{(\eta_2 \bar{\eta}_1)} \left((\dot{x}_{1\nu} + \dot{x}_{3\nu}) \Gamma^{\nu\tau\tau} + \dot{x}_{2\nu} \dot{x}_{1\lambda} \dot{x}_{3\sigma} \Gamma^{\nu\lambda\sigma} - \right. \right. \\ & \left. \left. - \dot{x}_{1\nu} \dot{x}_{3\lambda} \dot{x}_{2\sigma} \Gamma^{\nu\lambda\sigma} \right) + (\eta_1 \bar{\eta}_2) \dot{x}_{3\nu} \Gamma^{\tau\tau\nu} \right]. \end{aligned} \quad (5.38)$$

In the second equality in (5.38) we have identified all the terms which differ by a permutation of τ_2 with τ_3 , being trivially equivalent. We can distinguish two types of contributions: one containing only contracted three-point functions and the other where the three-point functions are saturated with three \dot{x}_i . The former can be rewritten in terms of the function $\Phi_{i,jk}$ by means of the relations (5.25) and we obtain

$$\begin{aligned} & i \int_0^L d\tau_1 \int_{-L}^0 d\tau_2 \int_{-L}^0 d\tau_3 \left[\frac{2}{(\eta_2 \bar{\eta}_1)} (\dot{x}_{1\nu} + \dot{x}_{3\nu}) \Gamma^{\nu\tau\tau} + (\eta_1 \bar{\eta}_2) \dot{x}_{3\nu} \Gamma^{\tau\tau\nu} \right] = \\ = & i \int_0^L d\tau_1 \int_{-L}^0 d\tau_2 \int_{-L}^0 d\tau_3 \left[\frac{2}{(\eta_2 \bar{\eta}_1)} \left(\frac{d}{d\tau_1} \Phi_{1,23} - \frac{d}{d\tau_2} \Phi_{1,23} - \frac{d}{d\tau_3} \Phi_{1,23} \right) + (\eta_1 \bar{\eta}_2) \frac{d}{d\tau_3} \Phi_{3,12} \right]. \end{aligned} \quad (5.39)$$

The divergent part of these integrals can be extracted from the table of integrals presented in app. C and we find

$$\frac{\Gamma^2(1/2 - \epsilon)}{16\pi^{3-2\epsilon}} \left[\frac{L^{4\epsilon}}{\cos \frac{\varphi}{2}} \left[\frac{2}{\epsilon} \log \left(\sec \left(\frac{\varphi}{2} \right) + 1 \right) - \frac{1}{2\epsilon^2} \right] + \frac{L^{4\epsilon}}{\epsilon} \cos \frac{\varphi}{2} \log \left(\cos \frac{\varphi}{2} \right) + O(1) \right]. \quad (5.40)$$

The procedure for determining the divergences of the latter contribution in (5.38) is more delicate, since we have to deal with the untraced three-point function. After a careful analysis, one gets

$$\begin{aligned} \frac{2i}{(\eta_2\bar{\eta}_1)} \int_0^L d\tau_1 \int_{-L}^0 d\tau_2 \int_{-L}^0 d\tau_3 \left(\dot{x}_{2\nu} \dot{x}_{1\lambda} \dot{x}_{3\sigma} \Gamma^{\nu\lambda\sigma} - \dot{x}_{1\nu} \dot{x}_{3\lambda} \dot{x}_{2\sigma} \Gamma^{\nu\lambda\sigma} \right) = \\ = -\frac{\Gamma^2(1/2-\epsilon)}{16\pi^{3-2\epsilon}} \frac{L^{4\epsilon}}{\epsilon} \cos \frac{\varphi}{2} \log \left(\cos \frac{\varphi}{2} \right) + O(1). \end{aligned} \quad (5.41)$$

If we sum (5.40) and (5.41), we can finally write down the result for $\mathcal{I}_{(a)} + \mathcal{I}_{(b)}$

$$\mathcal{I}_{(a)} + \mathcal{I}_{(b)} = \frac{\Gamma^2(1/2-\epsilon)}{16\pi^{3-2\epsilon}} \frac{L^{4\epsilon}}{\cos \frac{\varphi}{2}} \left[\frac{2}{\epsilon} \log \left(\sec \left(\frac{\varphi}{2} \right) + 1 \right) - \frac{1}{2\epsilon^2} \right] + O(1). \quad (5.42)$$

This completes the evaluation of the divergent part of all diagrams at two loops.

6 The final result: summing and renormalizing

In this section we shall add up the different diagrams which appear at two loops. Because we are actually working with an open contour, we have in principle two possibilities to perform this sum: we can take the trace $[\mathcal{W}_+$ in (2.16)] or the super-trace $[\mathcal{W}_-$ in (2.16)]. As we shall see, the first choice, that is the correct one for closed contours, appears to be consistent with an exponentiated form. We also discuss the renormalization of our result, paying particular attention to the peculiarities arising in three dimensions and in the presence of the exotic fermionic couplings.

6.1 Taking the trace

Let us consider the case of the trace. The bosonic bubble diagrams yield a four-dimensional-like contribution given by

$$\mathbb{B} = -g(\epsilon) \left(\cos \varphi - \cos^2 \frac{\theta}{2} \right) \frac{1}{\epsilon} \frac{\varphi}{\sin \varphi}, \quad (6.1)$$

where we have introduced the short-hand notation

$$g(\epsilon) = MN \left(\frac{2\pi}{\kappa} \right)^2 \frac{\Gamma^2(\frac{1}{2}-\epsilon)}{16\pi^{3-2\epsilon}} (\mu L)^{4\epsilon} \quad (6.2)$$

for future convenience. The fermionic bubble instead cancels when we take the trace, since it is odd in the exchange $N \leftrightarrow M$. The total result for the complete set of double-exchange diagrams is more elaborate and it can be usefully cast in the form

$$\begin{aligned} \mathbb{D} = 2g(\epsilon) \left[\frac{1}{\epsilon^2} \left[2 - \frac{3}{2} \frac{\cos \frac{\theta}{2}}{\cos \frac{\varphi}{2}} \right] + \frac{1}{\epsilon} \left[\frac{\cos \frac{\theta}{2}}{\cos \frac{\varphi}{2}} \left(4 \log \left(\sec \frac{\varphi}{2} + 1 \right) + \log \left(\cos \frac{\varphi}{2} \right) \right) + 1 \right] \right] + \\ + \frac{1}{4} \left[\frac{\cos \frac{\theta}{2}}{\cos \frac{\varphi}{2}} \left(\frac{1}{\epsilon} - 2 \log \left(1 + \sec \frac{\varphi}{2} \right) \right) \right]^2 - \frac{1}{2\epsilon} \cos^2 \frac{\theta}{2} \frac{\varphi}{\sin \varphi} + O(1). \end{aligned} \quad (6.3)$$

The diagrams which contain the gauge-fermion interaction yield instead the following result

$$\mathbb{V} = \frac{g(\epsilon)}{\epsilon^2} \left[\frac{\cos \frac{\theta}{2}}{\cos \frac{\varphi}{2}} - 2 \right] + \frac{g(\epsilon)}{\epsilon} \left[\varphi \cot \varphi - 2 \left(2 \frac{\cos \frac{\theta}{2}}{\cos \frac{\varphi}{2}} \log \left(\sec \frac{\varphi}{2} + 1 \right) + \log \left(\cos \frac{\varphi}{2} \right) + 1 \right) \right] + O(1). \quad (6.4)$$

We shall now sum these three contributions in order to obtain the unrenormalized value of \mathcal{W}_+ in (2.16) at two loops

$$\begin{aligned} [\mathcal{W}_+]_{2\text{-loop}} = & \frac{g(\epsilon)}{2\epsilon^2} \left(\frac{\cos \frac{\theta}{2}}{\cos \frac{\varphi}{2}} - 2 \right)^2 + \frac{2g(\epsilon)}{\epsilon} \frac{\cos \frac{\theta}{2}}{\cos \frac{\varphi}{2}} \left(2 - \frac{\cos \frac{\theta}{2}}{\cos \frac{\varphi}{2}} \right) \log \left(\sec \frac{\varphi}{2} + 1 \right) + \\ & + \frac{2g(\epsilon)}{\epsilon} \log \left(\cos \frac{\varphi}{2} \right) \left(\frac{\cos \frac{\theta}{2}}{\cos \frac{\varphi}{2}} - 1 \right) + O(1). \end{aligned} \quad (6.5)$$

In this expression the structure of the generalized potential is not manifest. Crucially we observe the presence of double-poles that are not expected to appear in the final expression of the generalized potential. In conventional Wilson loops, where only bosonic couplings are concerned, double-poles at two loops are simply understood as coming from the square of the one-loop result, by virtue of the non-abelian exponentiation theorem [44] (that holds even at renormalized level). The non-trivial contribution at second order in perturbation theory comes from the so-called maximally non-abelian part and in $\mathcal{N} = 4$ SYM, for example, involves crossed non-planar bosonic exchanges and interacting diagrams, stretching between the two lines. In our case, due to the presence of the fermionic couplings, we do not have an established exponentiation theorem at hand and we were forced to compute the full two-loop contribution to the quantum average. Incidentally, for our loops, double-poles appear both from exchange and interacting diagrams at variance with $\mathcal{N} = 4$ SYM, where non-abelian exponentiation forbids the presence of $1/\epsilon^2$ in vertex or bubble graphs. In order to proceed and extract a generalized potential, taking properly into account the one-loop and two-loop results, we need an exponentiation ansatz: we propose the following form for the unrenormalized loop

$$\mathcal{W}_+ = \frac{M \exp(V_N) + N \exp(V_M)}{N + M}. \quad (6.6)$$

It is not difficult to check that our results are compatible with this double-exponentiation where

$$\begin{aligned} V_N = & \left(\frac{2\pi}{\kappa} \right) N \left(\frac{\Gamma(1/2 - \epsilon)}{4\pi^{3/2 - \epsilon}} \right) (\mu L)^{2\epsilon} \left[\frac{1}{\epsilon} \left(\frac{\cos \frac{\theta}{2}}{\cos \frac{\varphi}{2}} - 2 \right) - 2 \frac{\cos \frac{\theta}{2}}{\cos \frac{\varphi}{2}} \log \left(\sec \left(\frac{\varphi}{2} \right) + 1 \right) \right] + \\ & + N^2 \left(\frac{2\pi}{\kappa} \right)^2 \frac{\Gamma^2(\frac{1}{2} - \epsilon)}{16\pi^{3 - 2\epsilon}} (\mu L)^{4\epsilon} \left[\frac{1}{\epsilon} \log \left(\cos^2 \frac{\varphi}{2} \right) \left(\frac{\cos \frac{\theta}{2}}{\cos \frac{\varphi}{2}} - 1 \right) + O(1) \right] \end{aligned} \quad (6.7)$$

The generalized potential V_M is of course obtained by exchanging M with N in the above formula. We remark that the actual exponentiation of the one-loop term is a non-trivial support of our assumption and of the correctness of our computations, involving a delicate balance between exchanging and interacting contributions. From the physical point of view

we could also justify the presence of two generalized potentials, simply recalling that we have two different test particles running in our contour. Following [40] it is straightforward to show that in $U(N) \times U(M)$ $\mathcal{N} = 6$ theories two kinds of particles arise from the relevant higgsing procedure and which transform respectively in the $(\mathbf{N}, \mathbf{1})$ and $(\mathbf{1}, \mathbf{M})$ representations and their conjugate, that we call W_N and W_M bosons. It is clear that a pair of W_N and W_M cannot form a singlet of the color indices and there is no generalization of the quark-antiquark potential in this case. On the other hand a pair of $W_N \bar{W}_N$ or $W_M \bar{W}_M$ do form color singlets, hence there are two potentials in this theory.

6.2 The renormalized generalized potentials

In this subsection we shall consider the problem of divergences and their renormalization for contour averages, examining first bosonic Wilson loops in general dimension D and then focussing the discussion to our case. We shall follow closely the original analysis of Polyakov, presented in his seminal paper [16].

Let us consider the first contribution in perturbation theory to the quantum average of the Wilson loop operator (we consider here the case of pure Yang-Mills theory with coupling constant g)

$$\mathcal{W}^{(1)} \simeq g^2 \oint_C \oint_C \frac{dx_\mu dx'^\mu}{[(x - x')^2]^{\frac{D-2}{2}}}. \quad (6.8)$$

For smooth contours the integral has a divergence when $D \geq 3$, arising at $x = x'$. We can brutally introduce an UV cut-off a^{-1}

$$\mathcal{W}^{(1)} \simeq g^2 \oint_C \oint_C \frac{dx_\mu dx'^\mu}{[(x - x')^2 + a^2]^{\frac{D-2}{2}}}, \quad (6.9)$$

taking care of the singularity at coincident points. To derive the singular behavior, we choose a parametrization where $(\dot{x})^2$ is a constant and we expand the integrand around the coincidence point, obtaining

$$[\mathcal{W}^{(1)}]_{sing.} \simeq g^2 \int_0^1 dt \int_t^1 ds \frac{\dot{x}^2(t)}{[\dot{x}^2(t)(s - t)^2 + a^2]^{\frac{D-2}{2}}}. \quad (6.10)$$

The integral can be easily evaluated, in the limit of $a \rightarrow 0$, giving the following leading contribution for $D > 3$

$$[\mathcal{W}^{(1)}]_{sing.} \simeq g^2 \frac{\sqrt{\pi}}{(a^2)^{\frac{D-3}{2}}} \frac{\Gamma(\frac{D-3}{2})}{\Gamma(\frac{D-2}{2})} L[C], \quad (6.11)$$

where $L[C] = \int_0^1 dt \sqrt{\dot{x}^2(t)}$ is the length of the contour. For $D = 4$ we recover the familiar result

$$[\mathcal{W}^{(1)}]_{sing.} \simeq g^2 \frac{\pi}{a} L[C], \quad (6.12)$$

interpreted as the one-loop mass renormalization for a test particle guided along the trajectory C . More generally, the Wilson loop can be renormalized multiplicatively [17]

$$\mathcal{W}(g) = \exp \left[-\hat{\gamma}(g_{ren.}) \frac{L[C]}{a} \right] \mathcal{W}(g_{ren.}), \quad (6.13)$$

where $W(g_{ren})$ is finite, provided we have inserted the renormalized coupling constant g_{ren} in its expression, while $\hat{\gamma}(g_{ren.})$ describes the perturbative mass renormalization of the test particle. On the other hand when cusps are present an additional subtraction, depending on the cusp angles, is mandatory and from the related renormalization constant one could extract the cusp anomalous dimension [17, 18]. In the case of *open* contours further gauge-dependent divergences could in general appear [45, 46], whose origin can be traced back to the self-energy corrections for the one-dimensional fermion living on the loop in the Gervais-Neveu description of the Wilson operator [47]. We remark that in four dimensions there is a precise distinction between the linear divergence (proportional to the length of the loop) and the cusp divergence (logarithmic in the cut-off).

In three dimensions the situation is more subtle: the expression (6.11) is singular in $D = 3$, due to the different infrared structure of three dimensional Yang-Mills theory. The integral (6.10) can be still evaluated, for $a \rightarrow 0$, obtaining

$$\mathcal{W}^{(1)} \simeq g^2 L[C] \log \frac{L[C]}{a}. \quad (6.14)$$

The divergence is no more power-like but it is logarithmic¹² in the cut-off: this could be expected because the Yang-Mills coupling constant has the dimension of a mass and we learn that, in general, we need some informations on the renormalization properties of the Wilson loop to properly single out cusp divergences.

We come now to discuss the renormalization of the cusped contour defining the generalized potential: in the maximal supersymmetric case, in four dimensions, the combined gauge-scalar propagator vanishes when attached to a single ray and the cusp anomalous divergence is easily extracted, at two-loops, from maximal non-abelian diagrams connecting the different lines [48]. We have seen, in the previous sections, that in the ABJ(M) case while bosonic diagrams do not produce singularities, when attached to a single line, fermionic propagators instead do: let us try to discuss the meaning of these unexpected divergences.

Starting from the expression (4.7) for the effective fermionic propagator and exploiting the fact that it can be written as a total derivative, a simple computation shows that the singular contribution to the one-loop result, for planar, closed circuits is contained in the integral

$$\int_0^1 dt |\dot{x}(t)| \left[\frac{1}{\eta(t)\bar{\eta}(0)} + \frac{1}{\eta(1)\bar{\eta}(t)} \right] \frac{1}{|x(0) - x(t)|}. \quad (6.15)$$

The singularity can be again extracted by introducing a cut-off a and, examining the behavior at the coincidence points, a short calculation produces

$$\log \left(\frac{L}{a} \right) \left[\frac{1}{\eta(1)\bar{\eta}(0)} + \frac{1}{\eta(1)\bar{\eta}(1)} + \frac{1}{\eta(0)\bar{\eta}(0)} + \frac{1}{\eta(1)\bar{\eta}(0)} \right], \quad (6.16)$$

with L being a length scale. We have taken into account the contribution coming from the two singular endpoints of the region of integration. For the supersymmetric BPS circle

¹²Thus it also shows up in dimensional regularisation at variance with the four dimensional case.

we realize immediately that, due to the anti-periodic boundary condition, this divergence is actually absent. For generic (not supersymmetric) choices of the fermionic couplings we expect instead a sort of mass renormalization for the test particles, logarithmic in the cut-off,

$$\frac{2\pi}{k} \log \frac{L[C]}{a}. \quad (6.17)$$

In the case of the cusped contour we have

$$\eta(0)\bar{\eta}(0) = \eta(1)\bar{\eta}(1) = 2i,$$

and both rays produce this kind of singularity: we interpret this divergence, present for the BPS straight line as well, as an *open* contour divergence [45], and we subtract it by defining a suitable renormalization constant Z_{open} , that multiplies our final result. Actually our generalized cusp, strictly speaking, is an open Wilson line and the gauge-invariant cusp divergence arises only after the aforementioned subtraction (ABJ theories are finite so no perturbative renormalization is needed for the Lagrangian).

At one-loop order, the simplest way to isolate the gauge-invariant content of our cusped Wilson loop is to subtract from (4.15) its value at $\varphi = \theta = 0$

$$\mathfrak{F}^{(1)} = \left(\frac{2\pi}{\kappa}\right) MN \left(\frac{\Gamma(1/2 - \epsilon)}{4\pi^{3/2-\epsilon}}\right) (\mu L)^{2\epsilon} \left[\frac{1}{\epsilon} \left(\frac{\cos \frac{\theta}{2}}{\cos \frac{\varphi}{2}} - 1 \right) - 2 \frac{\cos \frac{\theta}{2}}{\cos \frac{\varphi}{2}} \log \left(\sec \left(\frac{\varphi}{2} \right) + 1 \right) - \log 4 \right]. \quad (6.18)$$

One can easily realize that this prescription simply corresponds to the subtraction of the single-leg diagrams and, as a result of this renormalization, the 1/2 BPS straight line has expectation value 1 (at one-loop order). The same result can be obtained, more formally, by applying the general procedure presented in the seminal paper [18] (see their formula (26) at page 351). We notice that, with this subtraction, the singular one-loop contribution to the generalized cusped Wilson operator vanishes, as expected, at the supersymmetric point $\theta = \pm\varphi$, producing a first check of the correctness of our BPS configurations. Going to the second perturbative order, we notice that no further subtraction is needed to remove the spurious single-leg divergences, once the one-loop contribution has been properly treated: the two-loop potential in (6.7) vanishes at $\theta = \varphi = 0$. We are led therefore to define the renormalized expectation value \mathcal{W}_+^{ren} of our cusped loops from

$$\mathcal{W}_+ = Z_{open} \mathcal{W}_+^{ren}, \quad (6.19)$$

with

$$Z_{open} = \exp \left[- \left(\frac{2\pi}{\kappa} \right) N \left(\frac{\Gamma(\frac{1}{2} - \epsilon)}{4\pi^{3/2-\epsilon}} \right) (\mu L)^{2\epsilon} \left(\frac{1}{\epsilon} + \log 4 \right) \right]. \quad (6.20)$$

From these definitions we get the renormalized generalized potential $V_N^{\text{Ren.}}$

$$\begin{aligned} V_N^{\text{Ren.}} = & \left(\frac{2\pi}{\kappa} \right) N \left(\frac{\Gamma(\frac{1}{2} - \epsilon)}{4\pi^{3/2-\epsilon}} \right) (\mu L)^{2\epsilon} \left[\frac{1}{\epsilon} \left(\frac{\cos \frac{\theta}{2}}{\cos \frac{\varphi}{2}} - 1 \right) - 2 \frac{\cos \frac{\theta}{2}}{\cos \frac{\varphi}{2}} \log \left(\sec \left(\frac{\varphi}{2} \right) + 1 \right) + \log 4 \right] + \\ & + \left(\frac{2\pi}{\kappa} \right)^2 N^2 \left(\frac{\Gamma(\frac{1}{2} - \epsilon)}{4\pi^{3/2-\epsilon}} \right)^2 (\mu L)^{4\epsilon} \left[\frac{1}{\epsilon} \log \left(\cos \frac{\varphi}{2} \right)^2 \left(\frac{\cos \frac{\theta}{2}}{\cos \frac{\varphi}{2}} - 1 \right) + O(1) \right]. \end{aligned} \quad (6.21)$$

The quark-antiquark potential is recovered by taking the limit $\varphi \rightarrow \pi$ and following the prescription of [1]

$$V_N^{(s)}(R) = \frac{N}{k} \frac{1}{R} - \left(\frac{N}{k}\right)^2 \frac{1}{R} \log\left(\frac{T}{R}\right). \quad (6.22)$$

We observe a logarithmic, non-analytic term in T/R at the second non-trivial order that, as in four dimensions, is expected to disappear when resummation of the perturbative series is performed. We can also perform the opposite limit, taking large imaginary φ , and we recover the universal cusp anomaly

$$\gamma_{cusp} = \frac{N^2}{k^2}, \quad (6.23)$$

that is the result obtained directly from the light-like cusp [26].

7 Conclusions and outlook

In this paper we have studied a family of cusped Wilson loops in ABJ(M) super Chern-Simons theory, constructed from two $1/2$ BPS lines implying the presence of peculiar fermionic couplings [30]. They depend on two parameters, φ and θ , that describe the geometrical and R -symmetry angles, respectively, between the two rays. We have studied the supersymmetric properties of these configurations and their relation with closed contours, obtained through conformal transformations. Different limits on the parameters allow to reach interesting observables, as the analogous of the quark-antiquark potential or the universal cusp anomalous dimension. We have performed an explicit two-loop computation in dimensional regularization and we have obtained the divergent part of these contour operators. Our results suggest the existence of two generalized potentials in this theory and, after renormalization, we have obtained in the relevant limits the universal cusp anomaly and the $W_{N(M)}\bar{W}_{N(M)}$ binding energy.

The construction of a generalized potential from a cusped Wilson loop opens many interesting possibilities in ABJ(M) theory: one could try to compute the radiation of a particle moving along an arbitrary smooth path, as done in $\mathcal{N} = 4$ SYM [2]. Further, one could hope to find a three dimensional analogue of the set of TBA integral equations, recently discovered in [4, 5], describing non-perturbatively the D=4 generalized cusp (see also [49–51] for very recent developments). It is also tempting to speculate on the possibility to derive the infamous interpolating function $h(\lambda)$ [52–58], by comparing the integrability computations with exact results obtained through localization [35]. An important step in this program would be the derivation of general class of Wilson loops with lower degree of supersymmetry, specifically some analogue of the DGRT loops [20] in $\mathcal{N} = 6$ super Chern-Simons theory. A particular case, the wedge on S^2 , has been discussed here in sec. 3: a general construction of BPS loops on S^2 , preserving fractions of supersymmetry, will be presented soon [42]. It would be of course important to compute their quantum expectation value at weak coupling, by perturbation theory, and at strong coupling, using

string techniques. Hopefully their exact expression could be derived through localization methods.

Acknowledgements

This work was supported in part by the MIUR-PRIN contract 2009-KHZKRX. We warmly thank Antonio Bassetto, Valentina Cardinali, Valentina Forini, Valentina Giangreco Marotta Puletti and especially Nadav Drukker for useful discussions.

Appendices

A Basics of ABJ(M) action

Here we will collect some basic facts about the ABJ(M) action in Euclidean space-time. The gauge sector consists of two gauge fields $(A_\mu)_i^j$ and $(\hat{A}_\mu)_{\hat{i}}^{\hat{j}}$ belonging respectively to the adjoint of $U(N)$ and $U(M)$. The matter sector instead contains the complex fields $(C_I)_i^{\hat{i}}$ and $(\bar{C}^I)_{\hat{i}}^i$ as well as the fermions $(\psi_I)_i^{\hat{i}}$ and $(\bar{\psi}^I)_{\hat{i}}^i$. The fields $(C, \bar{\psi})$ transform in the $(\mathbf{N}, \bar{\mathbf{M}})$ of the gauge group $U(N) \times U(M)$ while the couple (\bar{C}, ψ) lives in the $(\bar{\mathbf{N}}, \mathbf{M})$. The additional capitol index $I = 1, 2, 3, 4$ belongs to the R -symmetry group $SU(4)$. In order to quantize the theory at the perturbative level, we have introduced the covariant gauge fixing function $\partial_\mu A^\mu$ for both gauge fields and two sets of ghosts (\bar{c}, c) and $(\bar{\hat{c}}, \hat{c})$. Therefore we work with the following Euclidian space action (see [24, 59, 60])

$$\begin{aligned} S_{\text{CS}} &= -i \frac{k}{4\pi} \int d^3x \epsilon^{\mu\nu\rho} \left[\text{Tr}(A_\mu \partial_\nu A_\rho + \frac{2}{3} A_\mu A_\nu A_\rho) - \text{Tr}(\hat{A}_\mu \partial_\nu \hat{A}_\rho + \frac{2}{3} \hat{A}_\mu \hat{A}_\nu \hat{A}_\rho) \right] \\ S_{\text{gf}} &= \frac{k}{4\pi} \int d^3x \left[\frac{1}{\xi} \text{Tr}(\partial_\mu A^\mu)^2 + \text{Tr}(\partial_\mu \bar{c} D_\mu c) - \frac{1}{\xi} \text{Tr}(\partial_\mu \hat{A}^\mu)^2 + \text{Tr}(\partial_\mu \bar{\hat{c}} D_\mu \hat{c}) \right] \\ S_{\text{Matter}} &= \int d^3x \left[\text{Tr}(D_\mu C_I D^\mu \bar{C}^I) + i \text{Tr}(\bar{\psi}^I \not{D} \psi_I) \right] + S_{\text{int}} \end{aligned} \quad (\text{A.1})$$

Here S_{int} consists of the sextic scalar potential and $\psi^2 C^2$ Yukawa type potentials spelled out in [24]. The matter covariant derivatives are defined as

$$\begin{aligned} D_\mu C_I &= \partial_\mu C_I + i(A_\mu C_I - C_I \hat{A}_\mu) \\ D_\mu \bar{C}^I &= \partial_\mu \bar{C}^I - i(\bar{C}^I A_\mu - \hat{A}_\mu \bar{C}^I) \\ D_\mu \psi_I &= \partial_\mu \psi_I + i(\hat{A}_\mu \psi_I - \psi_I A_\mu) \\ D_\mu \bar{\psi}^I &= \partial_\mu \bar{\psi}^I - i(\bar{\psi}^I \hat{A}_\mu - A_\mu \bar{\psi}^I). \end{aligned} \quad (\text{A.2})$$

B Feynman rules, useful perturbative results and some spinorology

FEYNMAN RULES: In the first part of this appendix we shall briefly review the Euclidean Feynman rules relevant for our computation and some general conventions. We use the position-space propagators, which are obtained from those in momentum space (see *e.g.* [31]) by means of the following master integral

$$\int \frac{d^{3-2\epsilon}p}{(2\pi)^{3-2\epsilon}} \frac{e^{ip \cdot x}}{(p^2)^s} = \frac{\Gamma(\frac{3}{2} - s - \epsilon)}{4^s \pi^{\frac{3}{2} - \epsilon} \Gamma(s)} \frac{1}{(x^2)^{\frac{3}{2} - s - \epsilon}}. \quad (\text{B.1})$$

In Landau gauge, for the gauge field propagators we find

$$\begin{aligned} \langle (A_\mu)_i^j(x) (A_\nu)_k^l(y) \rangle_0 &= \delta_i^l \delta_k^j \left(\frac{2\pi i}{\kappa} \right) \epsilon_{\mu\nu\rho} \partial_x^\rho \left(\frac{\Gamma(\frac{1}{2} - \epsilon)}{4\pi^{\frac{3}{2} - \epsilon}} \frac{1}{((x-y)^2)^{\frac{1}{2} - \epsilon}} \right), \\ \langle (\hat{A}_\mu)_{\hat{i}}^{\hat{j}}(x) (\hat{A}_\nu)_{\hat{k}}^{\hat{l}}(y) \rangle_0 &= -\delta_{\hat{i}}^{\hat{l}} \delta_{\hat{k}}^{\hat{j}} \left(\frac{2\pi i}{\kappa} \right) \epsilon_{\mu\nu\rho} \partial_x^\rho \left(\frac{\Gamma(\frac{1}{2} - \epsilon)}{4\pi^{\frac{3}{2} - \epsilon}} \frac{1}{((x-y)^2)^{\frac{1}{2} - \epsilon}} \right). \end{aligned} \quad (\text{B.2})$$

The scalar propagators are instead given by

$$\langle (C_I)_i^{\hat{j}}(x)(\bar{C}^J)_k^{\hat{l}}(y) \rangle_0 = \delta_I^J \delta_i^l \delta_k^{\hat{j}} \frac{\Gamma(\frac{1}{2} - \epsilon)}{4\pi^{\frac{3}{2} - \epsilon}} \frac{1}{((x-y)^2)^{\frac{1}{2} - \epsilon}} \equiv \delta_I^J \delta_i^l \delta_k^{\hat{j}} D(x-y). \quad (\text{B.3})$$

Finally we shall consider the case of the tree level fermionic two-point function

$$\langle (\psi_I)_i^{\hat{j}}(x)(\bar{\psi}^J)_k^{\hat{l}}(y) \rangle_0 = \delta_I^J \delta_i^{\hat{l}} \delta_k^{\hat{j}} i\gamma^\mu \partial_\mu \left(\frac{\Gamma(\frac{1}{2} - \epsilon)}{4\pi^{\frac{3}{2} - \epsilon}} \frac{1}{((x-y)^2)^{\frac{1}{2} - \epsilon}} \right). \quad (\text{B.4})$$

In our computation the interaction vertices will become relevant when considering either the one-loop correction to the propagators or the graphs containing the three-point functions. The one-loop two-point function for the gauge field was computed in [31] and in momentum space it is given by

$$\langle (A_\mu)_i^j(p)(A_\nu)_k^l(-p) \rangle_0^{\text{1loop}} = -\delta_i^l \delta_k^j \left(\frac{2\pi}{\kappa} \right)^2 M \frac{2^{4\epsilon-1} \pi^\epsilon \sec(\pi\epsilon)}{\Gamma(1-\epsilon)} (p^2)^{-\frac{1}{2}-\epsilon} \left(\delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} \right). \quad (\text{B.5})$$

In coordinate space it takes the form

$$\langle (A_\mu)_i^j(x)(A_\nu)_k^l(y) \rangle_0^{\text{1loop}} = \delta_i^l \delta_k^j \left(\frac{2\pi}{\kappa} \right)^2 \frac{M\Gamma^2(\frac{1}{2} - \epsilon)}{4\pi^{3-2\epsilon}} \left(\frac{\delta_{\mu\nu}}{((x-y)^2)^{1-2\epsilon}} - \partial_\mu \partial_\nu \left(\frac{((x-y)^2)^\epsilon}{4\epsilon(1+2\epsilon)} \right) \right). \quad (\text{B.6})$$

The correction to the gauge propagator of \hat{A} is very similar to (B.6): we have simply to replace M with N and $\delta_i^l \delta_k^j$ with $\delta_i^{\hat{l}} \delta_k^{\hat{j}}$.

Next one to consider the one-loop corrections to the fermion propagator. In momentum space it is given by

$$\langle (\psi_I)_i^{\hat{j}}(p)(\bar{\psi}^J)_k^{\hat{l}}(-p) \rangle_0^{\text{1loop}} = -2i\delta_i^{\hat{l}} \delta_k^{\hat{j}} (N-M) \frac{16^{\epsilon-1} \pi^\epsilon (p^2)^{-\frac{1}{2}-\epsilon} \sec(\pi\epsilon)}{\Gamma(1-\epsilon)}. \quad (\text{B.7})$$

Notice that this expression is finite when ϵ approaches zero. Its expression in coordinate space is then obtained by taking the Fourier-transform

$$\langle (\psi_I)_i^{\hat{j}}(x)(\bar{\psi}^J)_k^{\hat{l}}(y) \rangle_0^{\text{1loop}} = -i\delta_i^{\hat{l}} \delta_k^{\hat{j}} (N-M) \frac{\Gamma^2(\frac{1}{2} - \epsilon)}{16\pi^{3-2\epsilon}} \frac{1}{((x-y)^2)^{1-2\epsilon}}. \quad (\text{B.8})$$

The last ingredient that is necessary for our analysis of the two-loop behavior of the cusp in ABJ(M) is the integral

$$\begin{aligned} \Gamma^{\lambda\mu\nu}(x_1, x_2, x_3) &= \left(\frac{\Gamma(\frac{1}{2} - \epsilon)}{4\pi^{3/2-\epsilon}} \right)^3 \partial_{x_1^\lambda} \partial_{x_2^\mu} \partial_{x_3^\nu} \int \frac{d^{3-2\epsilon}w}{(x_{1w}^2)^{1/2-\epsilon} (x_{2w}^2)^{1/2-\epsilon} (x_{3w}^2)^{1/2-\epsilon}} \equiv \\ &\equiv \partial_{x_1^\lambda} \partial_{x_2^\mu} \partial_{x_3^\nu} \Phi, \end{aligned} \quad (\text{B.9})$$

which governs all the three point functions appearing in our analysis. Actually, for planar loops we shall never need the closed form of (B.9) but only its value when two of the indices

are contracted

$$\Gamma^{\lambda\lambda\rho}(x_1, x_2, x_3) = \partial_{x_3^\rho}(\partial_{x_1} \cdot \partial_{x_2})\Phi = \frac{1}{2}\partial_{x_3^\rho}[\square_{x_3} - \square_{x_1} - \square_{x_2}]\Phi \equiv \partial_{x_3^\rho}\Phi_{3,12}, \quad (\text{B.10a})$$

$$\Gamma^{\lambda\rho\lambda}(x_1, x_2, x_3) = \partial_{x_2^\rho}(\partial_{x_1} \cdot \partial_{x_3})\Phi = \frac{1}{2}\partial_{x_2^\rho}[\square_{x_2} - \square_{x_1} - \square_{x_3}]\Phi \equiv \partial_{x_2^\rho}\Phi_{2,13}, \quad (\text{B.10b})$$

$$\Gamma^{\rho\lambda\lambda}(x_1, x_2, x_3) = \partial_{x_1^\rho}(\partial_{x_2} \cdot \partial_{x_3})\Phi = \frac{1}{2}\partial_{x_1^\rho}[\square_{x_1} - \square_{x_2} - \square_{x_3}]\Phi \equiv \partial_{x_1^\rho}\Phi_{1,23}, \quad (\text{B.10c})$$

where we took advantage of the invariance of the scalar function Φ under translations $[(\partial_{x_1^\lambda} + \partial_{x_2^\lambda} + \partial_{x_3^\lambda})\Phi = 0]$ and introduced the short-hand notation

$$\Phi_{i,jk} = -\frac{\Gamma^2(1/2 - \epsilon)}{32\pi^{3-2\epsilon}} \left[\frac{1}{(x_{ij}^2)^{\frac{1}{2}-\epsilon}(x_{ik}^2)^{\frac{1}{2}-\epsilon}} - \frac{1}{(x_{ij}^2)^{\frac{1}{2}-\epsilon}(x_{kj}^2)^{\frac{1}{2}-\epsilon}} - \frac{1}{(x_{ik}^2)^{\frac{1}{2}-\epsilon}(x_{jk}^2)^{\frac{1}{2}-\epsilon}} \right]. \quad (\text{B.11})$$

In our computation we are also led to consider the value of $\Phi_{i,jk}$ at coincident points. For $\epsilon > 1/2$ they are finite and given by

$$\Phi_{i,ik} = \frac{1}{2} \left(\frac{\Gamma(1/2 - \epsilon)}{4\pi^{3/2-\epsilon}} \right)^2 \frac{1}{(x_{ik}^2)^{1-2\epsilon}}, \quad \Phi_{i,jj} = -\frac{1}{2} \left(\frac{\Gamma(1/2 - \epsilon)}{4\pi^{3/2-\epsilon}} \right)^2 \frac{1}{(x_{ij}^2)^{1-2\epsilon}}. \quad (\text{B.12})$$

In the spirit of dimensional regularization we extend these result to any value of ϵ ¹³.

The three point-function (B.9) for our specific choice of x_1, x_2 and x_3 obeys a set of useful identities. Consider for instance the case when x_2 and x_3 belong to the first edge of the cusp, while x_1 is located on the opposite one. Then we can introduce the three orthonormal vectors

$$\dot{x}_2^\nu, \quad v^\nu = \frac{1}{r}(\dot{x}_1^\nu - (\dot{x}_1 \cdot \dot{x}_2)\dot{x}_2^\nu), \quad n^\nu = \frac{1}{r}\epsilon^{\nu\alpha\beta}\dot{x}_{2\alpha}\dot{x}_{1\beta} \quad \left[r \equiv \sqrt{|\dot{x}_1|^2|\dot{x}_2|^2 - (\dot{x}_1 \cdot \dot{x}_2)^2} \right]. \quad (\text{B.13})$$

and we can write the trivial identity

$$v_\nu(\Gamma^{\tau\nu\tau} - \Gamma^{\tau\tau\nu}) - \dot{x}_{2\lambda}\dot{x}_{2\rho}v_\nu(\Gamma^{\lambda\nu\rho} - \Gamma^{\lambda\rho\nu}) = n_\lambda n_\rho v_\nu(\Gamma^{\lambda\nu\rho} - \Gamma^{\lambda\rho\nu}) = 0, \quad (\text{B.14})$$

which follows from the complexness condition $\delta_{\mu\nu} = \dot{x}_{2\mu}\dot{x}_{2\nu} + n_\mu n_\nu + v_\mu v_\nu$. If we use the explicit form of the vector v , (B.14) takes the form

$$(\dot{x}_{1\nu} - (\dot{x}_1 \cdot \dot{x}_2)\dot{x}_{2\nu})(\Gamma^{\tau\nu\tau} - \Gamma^{\tau\tau\nu}) = \dot{x}_{2\lambda}\dot{x}_{2\rho}\dot{x}_{1\nu}(\Gamma^{\lambda\nu\rho} - \Gamma^{\lambda\rho\nu}). \quad (\text{B.15})$$

FERMIONIC CONTRACTIONS: When computing the *fermionic diagrams* contributing to the Wilson loop defined by the super-connection (2.1) we often encounter bilinears constructed with the spinors η and $\bar{\eta}$ defined by the two relations

$$(\dot{x}^\mu \gamma_\mu)_\alpha{}^\beta = \frac{1}{2i}|\dot{x}|(\eta^\beta \bar{\eta}_\alpha + \eta_\alpha \bar{\eta}^\beta) \quad (\eta^\beta \bar{\eta}_\alpha - \eta_\alpha \bar{\eta}^\beta) = 2i\delta_\alpha^\beta, \quad (\text{B.16})$$

For instance, the most common is

$$\eta_1 \gamma^\mu \bar{\eta}_2, \quad (\text{B.17})$$

¹³ This is equivalent to the usual statement that massless tadpoles vanish in dimensional regularization.

where the superscripts 1 and 2 denote two different points on the contour. We can determine its value up to an overall factor by means of the following corollary of (B.16)

$$\bar{\eta}_\alpha \eta^\beta = i \left(\mathbb{1} + \frac{\dot{x}^\lambda \gamma_\lambda}{|\dot{x}|} \right)_\alpha^\beta. \quad (\text{B.18})$$

Consider in fact the product $(\eta_1 \bar{\eta}_2)(\eta_1 \gamma^\mu \bar{\eta}_2)$. We can rewrite it as

$$(\eta_1 \bar{\eta}_2)(\eta_1 \gamma^\mu \bar{\eta}_2) = \bar{\eta}_{2\alpha} \eta_2^\beta (\gamma^\mu)_\beta{}^\rho \bar{\eta}_{1\rho} \eta_1^\alpha = -\text{Tr} \left[\left(\mathbb{1} + \frac{\dot{x}_2^\lambda \gamma_\lambda}{|\dot{x}_2|} \right) \gamma^\mu \left(\mathbb{1} + \frac{\dot{x}_1^\nu \gamma_\nu}{|\dot{x}_1|} \right) \right], \quad (\text{B.19})$$

where we used (B.18) in order to eliminate the spinors from the expression. Thus

$$(\eta_2 \gamma^\mu \bar{\eta}_1) = -\frac{2}{(\eta_1 \bar{\eta}_2)} \left[\frac{\dot{x}_1^\mu}{|\dot{x}_1|} + \frac{\dot{x}_2^\mu}{|\dot{x}_2|} - i \frac{\dot{x}_2^\lambda}{|\dot{x}_2|} \frac{\dot{x}_1^\nu}{|\dot{x}_1|} \epsilon_{\lambda\nu}{}^\mu \right]. \quad (\text{B.20})$$

The only undetermined factor in (B.20) is the scalar contraction $(\eta_1 \bar{\eta}_2)$. The condition (B.18) however determines its *norm*, i.e the product $(\eta_1 \bar{\eta}_2)(\eta_2 \bar{\eta}_1)$

$$(\eta_1 \bar{\eta}_2)(\eta_2 \bar{\eta}_1) = -\text{Tr} \left[\left(\mathbb{1} + \frac{\dot{x}_2^\lambda}{|\dot{x}_2|} \gamma_\lambda \right) \left(\mathbb{1} + \frac{\dot{x}_1^\nu}{|\dot{x}_1|} \gamma_\nu \right) \right] = -2 \left[1 + \frac{(\dot{x}_1 \cdot \dot{x}_2)}{|\dot{x}_1| |\dot{x}_2|} \right]. \quad (\text{B.21})$$

There is a second bilinear that will be relevant, namely the one containing three Dirac matrices

$$\eta_1 \gamma^\lambda \gamma^\mu \gamma^\nu \bar{\eta}_2. \quad (\text{B.22})$$

Its evaluation reduces to the previous case because of the following identity

$$\gamma^\rho \gamma^\mu \gamma^\sigma = \delta^{\rho\mu} \gamma^\sigma + \delta^{\mu\sigma} \gamma^\rho - \delta^{\rho\sigma} \gamma^\mu + i \epsilon^{\rho\mu\sigma} \mathbb{1}, \quad (\text{B.23})$$

which holds for three-dimensional Euclidean gamma matrices.

For completeness, we shall also give all the possible scalar contractions for our specific circuit

$$\begin{aligned} \eta_1 \bar{\eta}_1 &= 2i, & \eta_2 \bar{\eta}_2 &= 2i, & \eta_1 \bar{\eta}_2 &= 2ie^{i\delta} \cos \frac{\varphi}{2}, & \eta_2 \bar{\eta}_1 &= 2ie^{-i\delta} \cos \frac{\varphi}{2}, \\ \eta_2 \eta_1 &= -2ie^{-i\delta} \sin \frac{\varphi}{2}, & \bar{\eta}_1 \bar{\eta}_2 &= 2ie^{i\delta} \sin \frac{\varphi}{2}. \end{aligned} \quad (\text{B.24})$$

Here the indices 1 and 2 indicates the two different edges of the cusp.

C Perturbative integrals

In this appendix we have collected some results about the different integrals which we have encountered in our perturbative expansion of the cusp. In particular, below we show how to extract their divergent part, which is the relevant quantity in our calculation. We can naturally divide the integrals into two subfamilies: the double exchange and the vertex integrals.

C.1 Double exchange Integrals

First we consider the integrals of subsec. (5.2.1) appearing in the evaluation of the fermionic double exchange diagrams. When all propagators start and end on the same edge the integrations can be easily performed in terms of known functions. When at least one of the two propagators runs between the two edges, the procedure for extracting the divergent part in $1/\epsilon$ is more intricate. We consider first the diagram $[6.(b)]_{\text{up}}$, which is governed by the contour integral

$$\mathcal{D}_1 = \int_0^L d\tau_1 \int_{-L}^0 d\tau_2 (L + \tau_2)^{2\epsilon} (\partial_{\tau_2} - \partial_{\tau_1}) H(\tau_1, \tau_2), \quad (\text{C.1})$$

where we recall that $H(\tau_1, \tau_2) = (\tau_1^2 + \tau_2^2 - 2\tau_1\tau_2 \cos \varphi)^{-\frac{1}{2} + \epsilon}$. Since \mathcal{D}_1 is multiplied by $1/\epsilon$ in the expression for $[6.(b)]_{\text{up}}$, we are actually interested in its divergent and finite part. In order to compute them we first integrate by parts the derivative with respect to τ_2 and perform the integration of the derivative with respect to τ_1 . Then \mathcal{D}_1 takes the following form

$$\begin{aligned} \mathcal{D}_1 = & \int_0^L d\tau_1 L^{2\epsilon} (H(\tau_1, 0) - H(\tau_1, -L)) - 2\epsilon \int_0^L d\tau_1 \int_{-L}^0 d\tau_2 (L + \tau_2)^{2\epsilon-1} (H(\tau_1, \tau_2) - \\ & - H(\tau_1, -L)) - \int_{-L}^0 d\tau_2 (L + \tau_2)^{2\epsilon} (H(L, \tau_2) - H(0, \tau_2)). \end{aligned} \quad (\text{C.2})$$

The double integral in (C.2), which is multiplied by ϵ , is finite and thus we can drop it since it does not yield divergent or finite terms. For the two single integrals the relevant contribution is easily evaluated and it is given by

$$\mathcal{D}_1 = L^{4\epsilon} \left[\frac{1}{\epsilon} - 2 \log \left(\sec \left(\frac{\varphi}{2} \right) + 1 \right) + O(\epsilon) \right] \quad (\text{C.3})$$

Next we examine the contour integral controlling the graph $[7.(b)]_{\text{up}}$:

$$\mathcal{D}_2 = \int_0^L d\tau_1 \int_{-L}^0 d\tau_4 (-\tau_4)^{2\epsilon} (\partial_{\tau_4} - \partial_{\tau_1}) H(\tau_1, \tau_4). \quad (\text{C.4})$$

If we introduce the short-hand notation $G(\tau_4, \tau_1) = (\partial_{\tau_4} - \partial_{\tau_1}) H(\tau_1, \tau_4)$, we can rewrite the above integral as follows

$$\begin{aligned} \mathcal{D}_2 = & L^{4\epsilon} \int_0^1 d\tau_1 \int_0^1 d\tau_4 \tau_4^{2\epsilon} G(-\tau_4, \tau_1) = L^{4\epsilon} \int_0^1 d\tau_1 \int_0^{\tau_1} d\tau_4 [\tau_4^{2\epsilon} G(-\tau_4, \tau_1) + \tau_1^{2\epsilon} G(-\tau_1, \tau_4)] = \\ = & L^{4\epsilon} \int_0^1 d\tau_1 \int_0^1 d\tau_4 \tau_1 [(\tau_1 \tau_4)^{2\epsilon} G(-\tau_4 \tau_1, \tau_1) + \tau_1^{2\epsilon} G(-\tau_1, \tau_4 \tau_1)]. \end{aligned} \quad (\text{C.5})$$

If we now observe that $G(\tau_1, \tau_2)$ is an homogeneous function of degree $-2 + 2\epsilon$ in both variables, we can factor out the integration over τ_1

$$\mathcal{D}_2 = L^{4\epsilon} \int_0^1 d\tau_1 \int_0^1 d\tau_4 \tau_1^{-1+4\epsilon} [\tau_4^{2\epsilon} G(-\tau_4, 1) + G(-1, \tau_4)] = \frac{L^{4\epsilon}}{4\epsilon} \int_0^1 d\tau_4 (\tau_4^{2\epsilon} + 1) G(-1, \tau_4). \quad (\text{C.6})$$

Subsequently we expand in power of ϵ the integrand and perform the integration over τ_4 . We find

$$\mathcal{D}_2 = L^{4\epsilon} \left[\frac{1}{2\epsilon} + \log \left(\frac{1}{4} \cos \left(\frac{\varphi}{2} \right) \sec^4 \left(\frac{\varphi}{4} \right) \right) + O(\epsilon) \right]. \quad (\text{C.7})$$

The last non trivial integral is the one appearing in the diagram [7.(c)]_{up} and it is given by

$$\begin{aligned} \mathcal{D}_3 &= \int_0^L d\tau_1 \int_0^{\tau_1} d\tau_2 \int_{-L}^0 d\tau_3 \int_{-L}^{\tau_3} d\tau_4 (\partial_{\tau_1} - \partial_{\tau_3}) H(\tau_1, \tau_3) (\partial_{\tau_4} - \partial_{\tau_2}) H(\tau_2, \tau_4) = \\ &= \int_0^L d\tau_1 \int_0^{\tau_1} d\tau_2 \int_{-L}^0 d\tau_3 \int_{-L}^{\tau_3} d\tau_4 [\partial_{\tau_1} H(\tau_1, \tau_3) \partial_{\tau_4} H(\tau_2, \tau_4) + \partial_{\tau_3} H(\tau_1, \tau_3) \partial_{\tau_2} H(\tau_2, \tau_4) - \\ &\quad - \partial_{\tau_1} H(\tau_1, \tau_3) \partial_{\tau_2} H(\tau_2, \tau_4) - \partial_{\tau_3} H(\tau_1, \tau_3) \partial_{\tau_4} H(\tau_2, \tau_4)] \end{aligned} \quad (\text{C.8})$$

Consider the first two term in (C.8): we can perform two of the four integrations and we can write

$$\begin{aligned} L^{4\epsilon} \int_0^1 d\tau_1 \int_{-1}^0 d\tau_2 &\left[[H(1, \tau_2) - H(\tau_1, \tau_2)][H(\tau_1, \tau_2) - H(\tau_1, -1)] + \right. \\ &\quad \left. + [H(\tau_1, 0) - H(\tau_1, \tau_2)][H(\tau_1, \tau_2) - H(0, \tau_2)] \right]. \end{aligned} \quad (\text{C.9})$$

When expanding the integrand of (C.9) we encounter three basic divergent integrals, whose ϵ -expansion can be easily determined. The first one is very simple and yields

$$(\text{I})_{\text{C.9}} = \int_0^L d\tau_1 \int_{-L}^0 d\tau_2 H(\tau_1, 0) H(0, \tau_2) = \frac{L^{4\epsilon}}{4\epsilon^2}, \quad (\text{C.10})$$

The other two require a little effort. We have an integral which has the structure of the single exchange of a scalar in four dimensions up to the redefinitions $2\epsilon \rightarrow 4\epsilon$

$$\begin{aligned} (\text{II})_{\text{C.9}} &= \int_0^L d\tau_1 \int_{-L}^0 d\tau_2 H(\tau_1, \tau_2)^2 = \int_0^L d\tau_1 \int_{-L}^0 d\tau_2 \frac{1}{(\tau_1^2 + \tau_2^2 - 2\tau_1\tau_2 \cos \varphi)^{1-4\epsilon}} = \\ &= \frac{L^{4\epsilon}}{4\epsilon} \frac{\varphi}{\sin \varphi} + O(1). \end{aligned} \quad (\text{C.11})$$

Finally we have to consider

$$\begin{aligned} (\text{III})_{\text{C.9}} &= \int_0^L d\tau_1 \int_{-L}^0 d\tau_2 H(\tau_1, \tau_2) H(\tau_1, 0) = L^{4\epsilon} \int_0^1 d\tau_1 \int_0^1 d\tau_2 H(\tau_1, -\tau_2) \tau_1^{-1+2\epsilon} = \\ &= L^{4\epsilon} \int_0^1 d\tau_1 \int_0^{\tau_1} d\tau_2 H(\tau_1, -\tau_2) \tau_1^{-1+2\epsilon} + L^{4\epsilon} \int_0^1 d\tau_2 \int_0^{\tau_2} d\tau_1 H(\tau_1, -\tau_2) \tau_1^{-1+2\epsilon}. \end{aligned} \quad (\text{C.12})$$

If we now use that $H(\tau_1, \tau_2)$ it is an homogeneous function of degree $-1+2\epsilon$ we can easily perform one of the two integrations and we find

$$\begin{aligned} &= \frac{L^{4\epsilon}}{4\epsilon} \int_0^1 d\tau_2 H(1, -\tau_2) + \frac{L^{4\epsilon}}{4\epsilon} \int_0^1 d\tau_1 [H(\tau_1, -1) - 1] \tau_1^{-1+2\epsilon} + \frac{L^{4\epsilon}}{4\epsilon} \int_0^1 d\tau_1 \tau_1^{-1+2\epsilon} = \\ &= -\frac{L^{4\epsilon}}{2\epsilon} \log \left(\cos \frac{\varphi}{2} \right) + \frac{L^{4\epsilon}}{8\epsilon^2} + O(1). \end{aligned} \quad (\text{C.13})$$

Then the divergent part of the integral (C.9) is obtained by considering the combination

$$-(\text{I})_{\text{C.9}} - 2(\text{II})_{\text{C.9}} + 2(\text{III})_{\text{C.9}} = -\frac{L^{4\epsilon}}{2\epsilon} \frac{\varphi}{\sin \varphi} - \frac{L^{4\epsilon}}{\epsilon} \log \left(\cos \frac{\varphi}{2} \right) + O(1). \quad (\text{C.14})$$

The last two terms in (C.8) can be shown to be equal by performing the change of variables $\tau_1 \leftrightarrow -\tau_4$ and $\tau_2 \leftrightarrow -\tau_3$. Thus we double the first one and we perform one of the four integration. We obtain

$$\mathcal{D}_4 \equiv -2L^{4\epsilon} \int_0^1 d\tau_1 \int_{-1}^0 d\tau_3 \int_{-1}^{\tau_3} d\tau_4 \partial_{\tau_1} H(\tau_1, \tau_3) [H(\tau_1, \tau_4) - H(0, \tau_4)] \quad (\text{C.15})$$

This integral is quite hard to compute and it will also appear when consider the vertex diagrams. To evaluate it, we shall introduce the auxiliary function $F(\tau_1, \tau_3) \equiv H(\tau_1, \tau_3) - H(0, \tau_3)$, which obeys the following relation

$$\tau_1 \partial_{\tau_1} F(\tau_1, \tau_3) + \tau_3 \partial_{\tau_3} F(\tau_1, \tau_3) - (2\epsilon - 1)F(\tau_1, \tau_3) = 0, \quad (\text{C.16})$$

since it is a homogeneous of degree $(2\epsilon - 1)$. Next we use (C.16) to eliminate $\partial_{\tau_1} H(\tau_1, \tau_3) [\equiv \partial_{\tau_1} F(\tau_1, \tau_3)]$ from (C.15) and we find

$$\begin{aligned} \frac{\mathcal{D}_4}{2} &= -(2\epsilon - 1)L^{4\epsilon} \int_0^1 \frac{d\tau_1}{\tau_1} \int_{-1}^0 d\tau_3 \int_{-1}^{\tau_3} d\tau_4 F(\tau_1, \tau_3) F(\tau_1, \tau_4) + \\ &+ L^{4\epsilon} \int_0^1 \frac{d\tau_1}{\tau_1} \int_{-1}^0 d\tau_3 \int_{-1}^{\tau_3} d\tau_4 F(\tau_1, \tau_4) \tau_3 \partial_{\tau_3} F(\tau_1, \tau_3) = -\frac{(2\epsilon - 1)L^{4\epsilon}}{2} \int_0^1 \frac{d\tau_1}{\tau_1} \left(\int_{-1}^0 d\tau_4 F(\tau_1, \tau_4) \right)^2 - \\ &- L^{4\epsilon} \int_0^1 \frac{d\tau_1}{\tau_1} \int_{-1}^0 d\tau_4 \tau_4 F(\tau_1, \tau_4) F(\tau_1, \tau_4) - \frac{L^{4\epsilon}}{2} \int_0^1 \frac{d\tau_1}{\tau_1} \left(\int_{-1}^0 d\tau_4 F(\tau_1, \tau_4) \right)^2 = \\ &= -\epsilon L^{4\epsilon} \int_0^1 \frac{d\tau_1}{\tau_1} \left(\int_{-1}^0 d\tau_4 F(\tau_1, \tau_4) \right)^2 - L^{4\epsilon} \int_0^1 \frac{d\tau_1}{\tau_1} \int_{-1}^0 d\tau_4 \tau_4 F^2(\tau_1, \tau_4), \end{aligned} \quad (\text{C.17})$$

where we have performed the integration by parts over τ_3 . We shall now consider the two integrals in (C.17): the former,

$$V \equiv \int_0^1 \frac{d\tau_1}{\tau_1} \left(\int_{-1}^0 d\tau_4 F(\tau_1, \tau_4) \right)^2, \quad (\text{C.18})$$

can be simplified by recalling the relation (C.16) obeyed by $F(\tau_1, \tau_4)$. In fact

$$\begin{aligned} V &= \frac{1}{2\epsilon - 1} \int_0^1 \frac{d\tau_1}{\tau_1} \left(\int_{-1}^0 d\tau_4 \tau_1 \partial_{\tau_1} F(\tau_1, \tau_4) + \tau_4 \partial_{\tau_4} F(\tau_1, \tau_4) \right) \left(\int_{-1}^0 d\tau_3 F(\tau_1, \tau_3) \right) = \\ &= \frac{1}{2(2\epsilon - 1)} \left[\int_0^1 d\tau_1 \partial_{\tau_1} \left(\int_{-1}^0 d\tau_4 F(\tau_1, \tau_4) \right)^2 + 2 \int_0^1 \frac{d\tau_1}{\tau_1} \int_{-1}^0 d\tau_4 \tau_4 \partial_{\tau_4} F(\tau_1, \tau_4) \int_{-1}^0 d\tau_3 F(\tau_1, \tau_3) \right] = \\ &= \frac{1}{2(2\epsilon - 1)} \left(\int_{-1}^0 d\tau_4 F(1, \tau_4) \right)^2 + \frac{1}{2\epsilon - 1} \int_0^1 \frac{d\tau_1}{\tau_1} F(\tau_1, -1) \left(\int_{-1}^0 d\tau_4 F(\tau_1, \tau_4) \right) - \frac{V}{2\epsilon - 1}, \end{aligned}$$

where we have performed the integration over τ_1 in the first integral and we have integrated the second one by parts with respect to τ_4 . This is an equation that can be solved for V

and we get

$$V = \frac{1}{4\epsilon} \left(\int_{-1}^0 d\tau_4 F(1, \tau_4) \right)^2 + \frac{1}{2\epsilon} \int_0^1 \frac{d\tau_1}{\tau_1} \int_0^1 d\tau_4 F(\tau_1, -1) F(\tau_1, -\tau_4) \quad (\text{C.19})$$

In the same way we can show that

$$W \equiv \int_0^1 \frac{d\tau_1}{\tau_1} \int_{-1}^0 d\tau_4 \tau_4 [F(\tau_1, \tau_4)]^2 \quad (\text{C.20})$$

obeys the following relation

$$W = \frac{1}{4\epsilon} \int_{-1}^0 d\tau_4 \tau_4 [F(1, \tau_4)]^2 - \frac{1}{4\epsilon} \int_0^1 \frac{d\tau_1}{\tau_1} [F(\tau_1, -1)]^2. \quad (\text{C.21})$$

Collecting the two contributions V and W , we finally obtain

$$\begin{aligned} \frac{\mathcal{D}_4}{2} = & -L^{4\epsilon} \left[\frac{1}{4} \left(\int_{-1}^0 d\tau_4 F(1, \tau_4) \right)^2 + \frac{1}{2} \int_0^1 \frac{d\tau_1}{\tau_1} \int_0^1 d\tau_4 F(\tau_1, -1) F(\tau_1, -\tau_4) - \right. \\ & \left. - \frac{1}{4\epsilon} \int_0^1 d\tau_1 \tau_1 [F(1, -\tau_1)]^2 - \frac{1}{4\epsilon} \int_0^1 \frac{d\tau_1}{\tau_1} [F(\tau_1, -1)]^2 \right]. \end{aligned} \quad (\text{C.22})$$

The final step is to evaluate the four integrals in (C.22). The first one must be computed up to terms which vanish when $\epsilon \rightarrow 0$ and it yields

$$\int_{-1}^0 d\tau_4 F(1, \tau_4) = -\frac{1}{2\epsilon} + \log \left(\sec \frac{\varphi}{2} + 1 \right) + O(\epsilon). \quad (\text{C.23})$$

For the remaining three integrals, it is sufficient to determine the divergent part and we have

$$\int_0^1 \frac{d\tau_1}{\tau_1} \int_0^1 d\tau_4 F(\tau_1, -1) F(\tau_1, -\tau_4) = \frac{1}{2\epsilon} \log \left(2 \cos^2 \frac{\varphi}{4} \cos \frac{\varphi}{2} \right) + O(1) \quad (\text{C.24a})$$

$$\begin{aligned} \frac{1}{4\epsilon} \int_0^1 d\tau_1 \tau_1 [F(1, -\tau_1)]^2 &= \frac{1}{16\epsilon^2} - \frac{1}{8\epsilon} \varphi \cot \varphi - \frac{1}{2\epsilon} \log \left(\sec \left(\frac{\varphi}{2} \right) + 1 \right) + \\ &+ \frac{1}{8\epsilon} \log(2 \cos(\varphi) + 2) + O(1) \end{aligned} \quad (\text{C.24b})$$

$$\frac{1}{4\epsilon} \int_0^1 \frac{d\tau_1}{\tau_1} [F(\tau_1, -1)]^2 = \frac{1}{4\epsilon} \log \left(2 \cos^4 \frac{\varphi}{4} \cos \frac{\varphi}{2} \right) - \frac{1}{8\epsilon} \varphi \cot \varphi + O(1) \quad (\text{C.24c})$$

When collecting these different contributions, we finally find

$$\mathcal{D}_4 = \frac{L^{4\epsilon}}{2\epsilon} \left[\log \left(\cos \frac{\varphi}{2} \right) - \frac{1}{2} \varphi \cot \varphi \right] + O(1) \quad (\text{C.25})$$

C.2 Vertex Integrals

To begin with, we shall determine the divergent part of the integral in (5.29). We have two contributions: one has the form of the total derivative, while the second one is similar to

the integral \mathcal{D}_4 encountered in the previous subsection. First we consider

$$\begin{aligned} \int_0^L d\tau_1 \int_0^{\tau_1} d\tau_2 \int_{-L}^0 d\tau_3 \frac{d}{d\tau_3} \Phi_{1,23} &= \int_0^L d\tau_1 \int_0^{\tau_1} d\tau_2 (\Phi_{1,20} - \Phi_{1,2-L}) = \\ &= -\frac{\Gamma^2(\frac{1}{2} - \epsilon)}{32\pi^{3-2\epsilon}} \int_0^L d\tau_1 \int_0^{\tau_1} d\tau_2 ((\tau_1 - \tau_2)^{-1+2\epsilon} \tau_1^{-1+2\epsilon} - \tau_1^{-1+2\epsilon} \tau_2^{-1+2\epsilon} - (\tau_1 - \tau_2)^{-1+2\epsilon} \tau_2^{-1+2\epsilon}) + \\ &+ O(1) = \frac{\Gamma^2(\frac{1}{2} - \epsilon)}{32\pi^{3-2\epsilon}} \frac{L^{4\epsilon}}{4\epsilon^2} + O(1), \end{aligned} \quad (\text{C.26})$$

where we have neglected $\Phi_{1,2-L}$ since it does not yield divergent contributions. Next we consider the master integral

$$\int_0^1 d\tau_1 \int_0^{\tau_1} d\tau_2 \int_{-1}^0 d\tau_3 \frac{1}{(x_{13}^2)^{1/2-\epsilon}} \frac{d}{d\tau_3} \frac{1}{(x_{23}^2)^{1/2-\epsilon}}. \quad (\text{C.27})$$

With the help of an elementary change of variable, we can relate this integral to \mathcal{D}_4 and we find

$$\begin{aligned} \frac{\mathcal{D}_4}{2} - \int_0^L d\tau_1 \int_{-L}^0 d\tau_2 \int_{-L}^{\tau_2} \frac{d\tau_3}{(-\tau_3)^{1-2\epsilon}} \frac{d}{d\tau_1} H(\tau_1, \tau_2) &= \\ &= \left[\frac{L^{4\epsilon}}{2\epsilon} \left(\log \left(\cos \frac{\varphi}{2} \right) - \frac{1}{2} \varphi \cot \varphi \right) + \frac{L^{4\epsilon}}{8\epsilon^2} + O(1) \right] \end{aligned} \quad (\text{C.28})$$

We come now to discuss the two integrals containing only the traced three point function and appearing in the sum $\mathcal{I}_{(a)} + \mathcal{I}_{(b)}$ in (5.39). We first evaluate

$$\begin{aligned} \int_0^L d\tau_1 \int_{-L}^0 d\tau_2 \int_{-L}^0 d\tau_3 \frac{d}{d\tau_3} \Phi_{3,12} &= \int_0^L d\tau_1 \int_{-L}^0 d\tau_2 (\Phi_{0,12} - \Phi_{-L,12}) = \\ &= -\frac{\Gamma^2(1/2 - \epsilon)}{32\pi^{3-2\epsilon}} L^{4\epsilon} \int_0^1 d\tau_1 \int_0^1 d\tau_2 \left[\frac{1}{\tau_1^{1-2\epsilon} \tau_2^{1-2\epsilon}} - \frac{2\tau_1^{-1+2\epsilon}}{(\tau_1^2 + \tau_2^2 + 2\tau_1\tau_2 \cos \varphi)^{1/2-\epsilon}} \right] + O(1) = \\ &= -\frac{\Gamma^2(1/2 - \epsilon)}{32\pi^{3-2\epsilon}} \frac{L^{4\epsilon}}{\epsilon} \log \left(\cos \frac{\varphi}{2} \right) + O(1). \end{aligned} \quad (\text{C.29})$$

Again the sector of the integrand which is evaluated in $-L$ ($\Phi_{-L,12}$) yields a finite integral. Subsequently we determine the divergent part of

$$\begin{aligned} \int_0^L d\tau_1 \int_{-L}^0 d\tau_2 \int_{-L}^0 d\tau_3 \left(\frac{d}{d\tau_1} \Phi_{1,23} - \frac{d}{d\tau_3} \Phi_{1,23} - \frac{d}{d\tau_2} \Phi_{1,23} \right) &= \\ &= \int_0^L d\tau_1 \int_{-L}^0 d\tau_2 \int_{-L}^0 d\tau_3 \left(\frac{d}{d\tau_1} \Phi_{1,23} - 2 \frac{d}{d\tau_3} \Phi_{1,23} \right) = \\ &= \int_{-L}^0 d\tau_2 \int_{-L}^0 d\tau_3 (\Phi_{L,23} - \Phi_{0,23}) - 2 \int_0^L d\tau_1 \int_{-L}^0 d\tau_2 (\Phi_{1,20} - \Phi_{1,2-L}) \end{aligned} \quad (\text{C.30})$$

We have four different contributions in the above integral and we shall compute them separately

$$\begin{aligned}
(\text{I})_{\text{C.30}} &= \int_{-L}^0 d\tau_2 \int_{-L}^0 d\tau_3 \Phi_{L,23} = \\
&= \frac{\Gamma^2(1/2 - \epsilon)}{32\pi^{3-2\epsilon}} L^{4\epsilon} \int_{-1}^0 d\tau_2 \int_{-1}^0 d\tau_3 \left[\frac{((\tau_2 - \tau_3)^2)^{-\frac{1}{2} + \epsilon}}{(\tau_2^2 + 1 - 2\tau_2 \cos \varphi)^{\frac{1}{2} - \epsilon}} + 2 \leftrightarrow 3 \right] + O(1) = \\
&= \frac{L^{4\epsilon}}{\epsilon} \frac{\Gamma^2(1/2 - \epsilon)}{16\pi^{3-2\epsilon}} \log \left(\sec \left(\frac{\varphi}{2} \right) + 1 \right) + O(1)
\end{aligned} \tag{C.31a}$$

In (C.31a) we have first performed the integration over τ_3 in order to extract the divergence $1/\epsilon$. The remaining integral over τ_2 is finite and it can be computed at $\epsilon = 0$. We now come to consider

$$\begin{aligned}
(\text{II})_{\text{C.30}} &= \int_{-L}^0 d\tau_2 \int_{-L}^0 d\tau_3 \Phi_{0,23} = \\
&= -\frac{\Gamma^2(1/2 - \epsilon)}{32\pi^{3-2\epsilon}} \int_{-L}^0 d\tau_2 \int_{-L}^0 d\tau_3 \left[\frac{1}{\tau_2^{1-2\epsilon} \tau_3^{1-2\epsilon}} - ((\tau_2 - \tau_3)^2)^{-\frac{1}{2} + \epsilon} \left(\frac{1}{\tau_2^{1-2\epsilon}} + \frac{1}{\tau_3^{1-2\epsilon}} \right) \right] = \\
&= \frac{\Gamma^2(1/2 - \epsilon)}{32\pi^{3-2\epsilon}} \frac{L^{4\epsilon}}{2\epsilon^2} + O(1)
\end{aligned} \tag{C.31b}$$

The third integral is easy to compute since the φ -dependent contributions drop out from our calculation. In fact

$$\begin{aligned}
(\text{III})_{\text{C.30}} &= \int_0^L d\tau_1 \int_{-L}^0 d\tau_2 \Phi_{1,20} = \\
&= -\frac{\Gamma^2(1/2 - \epsilon)}{32\pi^{3-2\epsilon}} L^{4\epsilon} \int_0^1 d\tau_1 \int_0^1 d\tau_2 \left[\frac{\tau_1^{-1+2\epsilon}}{(\tau_1^2 + \tau_2^2 + 2\tau_1 \tau_2 \cos \varphi)^{\frac{1}{2} - \epsilon}} - (1 \leftrightarrow 2) - \tau_1^{-1+2\epsilon} \tau_2^{-1+2\epsilon} \right] = \\
&= \frac{\Gamma^2(1/2 - \epsilon)}{32\pi^{3-2\epsilon}} \frac{L^{4\epsilon}}{4\epsilon^2} + O(1)
\end{aligned} \tag{C.31c}$$

where the two contributions which are related by the exchange $\tau_1 \leftrightarrow \tau_2$ cancel.

$$\begin{aligned}
(\text{IV})_{\text{C.30}} &= \int_0^L d\tau_1 \int_{-L}^0 d\tau_2 \Phi_{1,2-L} = \\
&= \frac{\Gamma^2(1/2 - \epsilon)}{32\pi^{3-2\epsilon}} \int_0^L d\tau_1 \int_{-L}^0 d\tau_2 \left[(L + \tau_2)^{-1+2\epsilon} H(\tau_1, \tau_2) + H(\tau_1, -L)(L + \tau_2)^{-1+2\epsilon} \right] + O(1) = \\
&= 2 \frac{\Gamma^2(1/2 - \epsilon)}{32\pi^{3-2\epsilon}} L^{4\epsilon} \int_0^1 d\tau_1 \int_{-1}^0 d\tau_2 H(\tau_1, -1)(1 + \tau_2)^{-1+2\epsilon} + O(1) = \\
&= \frac{\Gamma^2(1/2 - \epsilon)}{32\pi^{3-2\epsilon}} \frac{L^{4\epsilon}}{\epsilon} \log \left(1 + \sec \left(\frac{\varphi}{2} \right) \right) + O(1)
\end{aligned} \tag{C.31d}$$

The integral over τ_1 can be computed at $\epsilon = 0$ since it is finite. In the above analysis we have consistently dropped all the integrals which do not produce a divergence. Then the

total result for the divergent part is

$$\begin{aligned} \int_0^L d\tau_1 \int_{-L}^0 d\tau_2 \int_{-L}^0 d\tau_3 \left(\frac{d}{d\tau_1} \Phi_{1,23} - \frac{d}{d\tau_3} \Phi_{1,23} - \frac{d}{d\tau_2} \Phi_{1,23} \right) &= (\text{I})_{\text{C.30}} - (\text{II})_{\text{C.30}} - \\ - 2(\text{III})_{\text{C.30}} + 2(\text{IV})_{\text{C.30}} &= \frac{\Gamma^2(1/2 - \epsilon)}{16\pi^{3-2\epsilon}} L^{4\epsilon} \left[\frac{2}{\epsilon} \log \left(\sec \left(\frac{\varphi}{2} \right) + 1 \right) - \frac{1}{2\epsilon^2} + O(1) \right]. \end{aligned} \quad (\text{C.32})$$

There is one remaining integral in the expansion of $\mathcal{I}_{(a)} + \mathcal{I}_{(b)}$ to be evaluated and it contains the three-point functions saturated with three \dot{x}_i . An (almost) straightforward computation leads to the following result

$$\begin{aligned} \int_0^L d\tau_1 \int_{-L}^0 d\tau_2 \int_{-L}^0 d\tau_3 [\dot{x}_{2\nu} \dot{x}_{1\lambda} \dot{x}_{3\sigma} \Gamma^{\nu\lambda\sigma} - \dot{x}_{1\nu} \dot{x}_{3\lambda} \dot{x}_{2\sigma} \Gamma^{\nu\lambda\sigma}] &= \\ = -\frac{\Gamma^2(1/2 - \epsilon)}{16\pi^{3-2\epsilon}} \frac{L^{4\epsilon}}{\epsilon} \left(\cos^2 \frac{\varphi}{2} \log \left(\cos \frac{\varphi}{2} \right) \right) + O(1). \end{aligned} \quad (\text{C.33})$$

The difference $\mathcal{I}_{(a)} - \mathcal{I}_{(b)}$ is instead governed by only one integral which contains traced three-point functions, namely

$$\begin{aligned} \int_0^L d\tau_1 \int_{-L}^0 d\tau_2 \int_{-L}^{\tau_2} d\tau_3 (\dot{x}_{1\nu} + \dot{x}_{2\nu}) (\Gamma^{\tau\tau\nu} - \Gamma^{\tau\nu\tau}) &= \int_0^L d\tau_1 \int_{-L}^0 d\tau_2 \int_{-L}^{\tau_2} d\tau_3 \left(\frac{d}{d\tau_3} \Phi_{3,12} - \frac{d}{d\tau_2} \Phi_{2,13} - \right. \\ &\quad \left. - \frac{d}{d\tau_1} \Phi_{3,12} \right) - \left(\frac{\Gamma(1/2 - \epsilon)}{4\pi^{3/2-\epsilon}} \right)^2 \int_0^L d\tau_1 \int_{-L}^0 d\tau_2 \int_{-L}^{\tau_2} d\tau_3 \frac{1}{(x_{13}^2)^{1/2-\epsilon}} \frac{d}{d\tau_1} \frac{1}{(x_{12}^2)^{1/2-\epsilon}}. \end{aligned} \quad (\text{C.34})$$

The last integral in (C.34) is equal to minus (C.27) and thus

$$\int_0^L d\tau_1 \int_{-L}^0 d\tau_2 \int_{-L}^{\tau_2} d\tau_3 \frac{1}{(x_{13}^2)^{1/2-\epsilon}} \frac{d}{d\tau_1} \frac{1}{(x_{12}^2)^{1/2-\epsilon}} = \left[\frac{L^{4\epsilon}}{2\epsilon} \left(\frac{\varphi}{2} \cot \varphi - \log \left(\cos \frac{\varphi}{2} \right) \right) - \frac{L^{4\epsilon}}{8\epsilon^2} + O(1) \right]. \quad (\text{C.35})$$

The remaining integrals can be then rearranged as follows

$$\int_0^L d\tau_1 \int_{-L}^0 d\tau_2 (2\Phi_{2,12} - \Phi_{-L,12} - \Phi_{0,12}) - \int_{-L}^0 d\tau_2 \int_{-L}^{\tau_2} d\tau_3 (\Phi_{3,L2} - \Phi_{3,02}) \quad (\text{C.36})$$

Consider first the contributions coming from $\Phi_{-L,12}$: the possible divergence arises when τ_2 approaches $-L$. On the other hand $\Phi_{-L,12}$ is regular when $\tau_2 \rightarrow -L$ and thus the integral of this quantity is finite. A similar analysis can be applied to $\Phi_{3,L2}$: the source of divergence is the region $\tau_3 \rightarrow \tau_2$, but the function is regular in this limit. Thus this term will not yield poles in ϵ . It remains to extract the divergent part of

$$\int_0^L d\tau_1 \int_{-L}^0 d\tau_2 (2\Phi_{2,12} - \Phi_{0,12}) + \int_{-L}^0 d\tau_2 \int_{-L}^{\tau_2} d\tau_3 \Phi_{3,02}, \quad (\text{C.37})$$

which for each term in (C.37) is given by

$$\begin{aligned} (\text{I})_{\text{C.37}} &= 2 \int_0^L d\tau_1 \int_{-L}^0 d\tau_2 \Phi_{2,12} = 2 \left(\frac{\Gamma(1/2 - \epsilon)}{4\pi^{3/2-\epsilon}} \right)^2 (\text{C.11}) = \left(\frac{\Gamma(1/2 - \epsilon)}{4\pi^{3/2-\epsilon}} \right)^2 \frac{L^{4\epsilon}}{4\epsilon} \frac{\varphi}{\sin \varphi} + O(1), \\ &\quad (\text{C.38a}) \end{aligned}$$

$$\begin{aligned}
(\text{II})_{\text{C.37}} &= \int_0^L d\tau_1 \int_{-L}^0 d\tau_2 \Phi_{0,12} = -\frac{L^{4\epsilon}}{2} \left(\frac{\Gamma(1/2 - \epsilon)}{4\pi^{3/2-\epsilon}} \right)^2 \int_0^1 d\tau_1 \int_0^1 d\tau_2 \left[\tau_1^{-1+2\epsilon} \tau_2^{-1+2\epsilon} - \right. \\
&\quad \left. - \frac{\tau_1^{-1+2\epsilon}}{(\tau_1^2 + \tau_2^2 + 2\tau_1\tau_2 \cos \varphi)^{\frac{1}{2}-\epsilon}} - \frac{\tau_2^{-1+2\epsilon}}{(\tau_1^2 + \tau_2^2 + 2\tau_1\tau_2 \cos \varphi)^{\frac{1}{2}-\epsilon}} \right] = \\
&= -\frac{L^{4\epsilon}}{2} \left(\frac{\Gamma(1/2 - \epsilon)}{4\pi^{3/2-\epsilon}} \right)^2 \frac{1}{\epsilon} \log \left(\cos \frac{\varphi}{2} \right) + O(1), \tag{C.38b}
\end{aligned}$$

$$(\text{III})_{\text{C.37}} = \int_{-L}^0 d\tau_2 \int_{-L}^{\tau_2} d\tau_3 \Phi_{3,02} = \frac{1}{2} \left(\frac{\Gamma(1/2 - \epsilon)}{4\pi^{3/2-\epsilon}} \right)^2 \frac{L^{4\epsilon}}{4\epsilon^2} + O(1). \tag{C.38c}$$

If we now collect all the contributions, we obtain a remarkably simple result

$$\int_0^L d\tau_1 \int_{-L}^0 d\tau_2 \int_{-L}^{\tau_2} d\tau_3 (\dot{x}_{1\nu} + \dot{x}_{2\nu}) (\Gamma^{\tau\tau\nu} - \Gamma^{\tau\nu\tau}) = \left(\frac{\Gamma(1/2 - \epsilon)}{4\pi^{3/2-\epsilon}} \right)^2 \frac{L^{4\epsilon}}{4\epsilon} \varphi \cot \frac{\varphi}{2} + O(1) \tag{C.39}$$

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